

# DISCRETE MAXIMAL FUNCTIONS IN HIGHER DIMENSIONS AND APPLICATIONS TO ERGODIC THEORY

MARIUSZ MIREK AND BARTOSZ TROJAN

**ABSTRACT.** We establish a higher dimensional counterpart of Bourgain's pointwise ergodic theorem along an arbitrary integer-valued polynomial mapping. We achieve this by proving variational estimates  $V_r$  on  $L^p$  spaces for all  $1 < p < \infty$  and  $r > \max\{p, p/(p-1)\}$ . Moreover, we obtain the estimates which are uniform in the coefficients of a polynomial mapping of fixed degree.

## 1. INTRODUCTION

In the mid 1980's Bourgain extended Birkhoff's pointwise ergodic theorem, proving that for any dynamical system  $(X, \mathcal{B}, \mu, T)$  on a  $\sigma$ -finite measure space  $X$  with an invertible measure preserving transformation  $T$  the averages along the squares

$$A_N f(x) = N^{-1} \sum_{n=1}^N f(T^{n^2} x)$$

converge  $\mu$ -almost everywhere on  $X$  for all  $f \in L^p(X, \mu)$  with  $p > 1$ , (see [2, 3]). Not long afterwards in [4], the squares were replaced by an arbitrary integer-valued polynomial. The restriction to the range  $p > 1$  in Bourgain's theorem turned out to be essential. Recently, Buczolich and Mauldin [5] have shown that the pointwise convergence of  $A_N f$  fails on  $L^1(X, \mu)$  (see also [20]).

In this article we are concerned with  $L^p(X, \mu)$  estimates for discrete higher dimensional analogues of the averaging operator and applications of such estimates to pointwise ergodic theorems.

Let  $(X, \mathcal{B}, \mu)$  be a  $\sigma$ -finite measure space with a family of invertible, commuting and measure preserving transformations  $T_1, T_2, \dots, T_{d_0}$  for some  $d_0 \in \mathbb{N}$ . Let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$  denote a polynomial mapping such that each  $\mathcal{P}_j$  is an integer-valued polynomial on  $\mathbb{Z}^k$  with  $\mathcal{P}_j(0) = 0$ . Define the averages

$$(1.1) \quad A_N^{\mathcal{P}} f(x) = N^{-k} \sum_{n \in \mathbb{N}_N^k} f(T_1^{\mathcal{P}_1(n)} T_2^{\mathcal{P}_2(n)} \dots T_{d_0}^{\mathcal{P}_{d_0}(n)} x)$$

where  $\mathbb{N}_N^k = \{1, 2, \dots, N\}^k$ . The results of this paper establish the following.

**Theorem A.** *Assume that  $p \in (1, \infty)$ . For every  $f \in L^p(X, \mu)$  there exists  $f^* \in L^p(X, \mu)$  such that*

$$\lim_{N \rightarrow \infty} A_N^{\mathcal{P}} f(x) = f^*(x)$$

*$\mu$ -almost everywhere on  $X$ .*

Classical proofs of pointwise convergence require  $L^p(X, \mu)$  bounds for maximal function, reducing the problem to proving pointwise convergence for a dense class of  $L^p(X, \mu)$  functions. However, establishing pointwise convergence on a dense class may be a difficult problem. For instance Bourgain's celebrated averaging operator along the squares is such an example. One of the possibilities, introduced by Bourgain

---

The authors were partially supported by NCN grant DEC-2012/05/D/ST1/00053.

in [4], for overcoming this problem is to control the  $r$ -variational seminorm  $V_r$  of a sequence of measurable functions  $(f_j : j \in \mathbb{N})$  defined by

$$V_r(f_j(x) : j \in \mathbb{N}) = \sup_{\substack{k_0 < k_1 < \dots < k_J \\ k_j \in \mathbb{N}}} \left( \sum_{j=1}^J |f_{k_j}(x) - f_{k_{j-1}}(x)|^r \right)^{1/r}.$$

Indeed, if  $V_r(f_j(x) : j \in \mathbb{N}) < \infty$  for some finite  $r \geq 1$  then the sequence  $(f_j(x) : j \in \mathbb{N})$  converges. Theorem A, in particular, will follow from more general result. Namely.

**Theorem B.** *Let  $p \in (1, \infty)$  and  $r > \max\{p, p/(p-1)\}$ . Then there is a constant  $C_{p,r} > 0$  such that for every  $f \in L^p(X, \mu)$*

$$(1.2) \quad \|V_r(A_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.$$

Moreover, the constant  $C_{p,r}$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

In view of Calderón's transference principle, one can reduce our problem and work on  $\mathbb{Z}^{d_0}$  rather than on an abstract measure space  $X$ . In this setting we consider the averages

$$(1.3) \quad M_N^{\mathcal{P}} f(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} f(x - \mathcal{P}(y))$$

for any finitely supported function  $f : \mathbb{Z}^{d_0} \rightarrow \mathbb{C}$ . We will be mainly interested in  $\ell^p$  bounds for  $r$ -variations of the averages  $M_N^{\mathcal{P}}$ . In this setup Theorem B can be reformulated in the following way.

**Theorem C.** *Let  $p \in (1, \infty)$  and  $r > \max\{p, p/(p-1)\}$ . There is a constant  $C_{p,r} > 0$  such that for every  $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$(1.4) \quad \|V_r(M_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

Moreover, the constant  $C_{p,r}$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

Theorem C is the main result of this article and generalizes recent one dimensional variational estimates of Krause [16]. However, its proof will strongly explore maximal theorem for  $M_N^{\mathcal{P}}$ . Namely, Theorem D which is the higher dimensional counterpart of Bourgain's theorem [4].

**Theorem D.** *For each  $p \in (1, \infty]$  there is a constant  $C_p > 0$  such that for every  $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$(1.5) \quad \left\| \sup_{N \in \mathbb{N}} |M_N^{\mathcal{P}} f| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

Moreover, the constant  $C_p$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

Bourgain's papers [2, 3, 4] initiated extensive study both in pointwise ergodic theory along various arithmetic subsets of the integers (see e.g. [1, 9, 11, 16, 23, 24, 39]) and investigations of discrete analogues of classical operators with arithmetic features (see e.g. [11, 10, 12, 13, 21, 22, 25, 27, 30, 32, 31, 34, 35, 36, 37]). Variational inequalities in harmonic analysis and ergodic theory have been the subject of many recent articles, see especially [14, 15, 16, 29, 40] and the references given there (see also [26, 28]).

We were motivated to study pointwise convergence of the averaging operators defined in (1.1) by recent results of Ionescu, Magyar, Stein and Wainger [11]. They considered pointwise convergence of some noncommutative variants of averaging operators along the polynomials of degree at most 2. The desire to better understand the restriction imposed on the degree of polynomials in [11], has led to, in particular, Theorem C and Theorem D from this paper. Furthermore, the recent paper of Krause [16] inspired us to study variational estimates in higher dimensions — see Theorem C — which in turn provide an approach to pointwise convergence different to the argument from [11]. Specifically, in this paper we relax the restriction on the degree of polynomials and we obtain all the results (maximal and variational estimates and pointwise convergence) for polynomials of arbitrary degree at the expense of the loss of the noncommutative setup which was the subject of [11].

The purpose of this article, compared with the prior works, is threefold. Firstly, as we said before, we shall relax the restriction for the degree of the polynomials from [11]. Secondly, we provide variational estimates and thirdly, we will establish bounds in the inequalities (1.4) and (1.5) which are uniform in the coefficients of underlying polynomial mapping. The last statement finds applications in the discrete multi-parameter theories of maximal functions and singular integral operators.

The inequality from (1.5) turned out to be decisive in one parameter theory, for instance in the ongoing project concerning  $\ell^p$  estimates for the maximal function corresponding with truncations of Radon transform from [13]. Namely, we have recently established, for some  $p > 1$ , the following inequality

$$(1.6) \quad \left\| \sup_{N \in \mathbb{N}} |T_N^{\mathcal{P}} f(x)| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}$$

where  $T_N^{\mathcal{P}} f$  is a truncated Radon transform along the polynomial mapping  $\mathcal{P}$ , i.e.

$$T_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}_N^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y),$$

where  $K$  is a Calderón-Zygmund kernel on  $\mathbb{R}^k$  and  $\mathbb{Z}_N^k = \{-N, \dots, -1, 0, 1, \dots, N\}^k$ . In fact, in the proof of inequality (1.6) we had to replace the supremum over the set of integers  $\mathbb{N}$  with the supremum over the set of dyadic numbers  $\{2^n : n \in \mathbb{N} \cup \{0\}\}$ . Since the operators  $T_N^{\mathcal{P}}$  are not positive we had to be more careful, but for  $N \in [2^n, 2^{n+1})$  we have the pointwise estimate

$$|T_N^{\mathcal{P}} f(x)| \leq C(|T_{2^n}^{\mathcal{P}} f(x)| + M_N^{\mathcal{P}} |f|(x))$$

for some  $C > 0$ .

The proof of Theorem D will be based on an idea of Ionescu and Wainger from [13] where they established  $\ell^p$  bounds for the discrete Radon transform by partitioning the operator into two parts, the first part controllable in  $\ell^p$  and the second part controllable in  $\ell^2$ . More precisely, for every  $\epsilon \in (0, 1]$  and  $\lambda > 0$  we are going to find an operator  $A_N^{\lambda, \epsilon}$  such that

$$\left\| \sup_{N \in \mathbb{N}} |M_N^{\mathcal{P}} f - A_N^{\lambda, \epsilon} f| \right\|_{\ell^2} \leq D_\epsilon \lambda^{-1} \|f\|_{\ell^2}$$

and for each  $p \in (1, \infty)$

$$\left\| \sup_{N \in \mathbb{N}} |A_N^{\lambda, \epsilon} f| \right\|_{\ell^p} \leq C_\epsilon \lambda^\epsilon \|f\|_{\ell^p}.$$

Then with the aid of these two estimates one can use restricted interpolation techniques as in [13] and conclude that (1.5) holds. The same idea was also explored in [11]. Here we are going to make use of this argument and provide a different approach to the estimates in  $\ell^2$  and  $\ell^p$  as compared both to Bourgain's paper [4] and Ionescu, Magyar, Stein and Wainger's paper [11]. Since the  $r$ -variational seminorm controls the supremum norm for any  $r \geq 1$  we only need  $r$ -variational estimates on  $\ell^2$ . The  $\ell^2$  theory for averaging operators along polynomials in [4] was built, to a large extent, on the circle method of Hardy and Littlewood and on the "logarithmic" lemma due to Bourgain (see [4], see also [18]).

**Bourgain's lemma.** Assume that  $\lambda_1 < \dots < \lambda_K \in \mathbb{R}$  and for  $j \in \mathbb{N}$  define the neighbourhoods

$$\mathcal{R}_j = \{\xi \in \mathbb{R} : \min_{1 \leq k \leq K} |\xi - \lambda_k| \leq 2^{-j}\}.$$

Then there exists a constant  $C > 0$  such that

$$\left\| \sup_{j \in \mathbb{N}} \left| \int_{\mathcal{R}_j} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \right| \right\|_{L^2(dx)} \leq C(\log K)^3 \|f\|_{L^2},$$

for every  $f \in L^2(\mathbb{R})$ .

Although Bourgain's lemma is interesting in its own right, and is a powerful tool in discrete problems, it has also found wide application in problems susceptible to time-frequency analysis (see e.g. [8, 19, 38]). Recently, Nazarov, Oberlin and Thiele [26] introduced a multi-frequency Calderón-Zygmund decomposition and extended Bourgain's estimates to provide  $L^p$  bounds and variational estimates (see also [28]). Some refinement of the results from [26] established by Krause [17] turned out to be an invaluable tool in variational estimates for Bourgain's averages along polynomials in [16]. In this paper we propose another approach which avoids the use of Bourgain's lemma or its counterparts.

Our approach to Theorem D and Theorem C proceeds in several stages. We begin with a particular lifting of the operator (1.3). This procedure will permit us to replace any polynomial mapping  $\mathcal{P}$  by a new polynomial mapping (the canonical polynomial mapping: see Lemma 3 in Section 2) which has all coefficients equal to 1. This will result in the uniform estimates and will reduce the study to the canonical polynomial mapping. In Section 3 we construct suitable approximating multipliers: see the definitions of (3.1), (3.17) and (3.20), and prove strong  $\ell^2$  bounds on their  $r$ -variations. These multipliers will be useful in proving Theorem D and Theorem C for  $p = 2$  in Section 4 and Section 5, respectively. The proofs of these  $\ell^2$  bounds, on the one hand, will be covered by the multi-dimensional variant of the circle method of Hardy and Littlewood.

On the other hand, we will make use of some elementary inequalities for  $r$ -variations from Section 2, which have not been used in this context. This is the novelty of the paper and allows us to study the approximating multipliers by a direct analysis which avoids using results like Bourgain's lemma.

In Section 3 we also provide the  $\ell^p$  theory,  $p > 1$ , necessary to obtain Theorem D. The strategy of the proof of  $\ell^p$  bounds will be very simple. We shall compare the discrete norm  $\|\cdot\|_{\ell^p}$  of our approximating multipliers with the continuous norm  $\|\cdot\|_{L^p}$  of certain multipliers which are a priori bounded on  $L^p$ . But this will only give good bounds when  $N$  is restricted to the large cubes depending on  $\lambda$  as in the Ionescu-Wainger partition. The small cubes will be covered by a restricted  $\ell^p$  bound with logarithmic loss for the operator (1.3): see Theorem 5. This idea was pioneered by Bourgain in [4] to prove the full range of  $\ell^p$  estimates. Here we will explore this idea, giving a slightly simpler proof of this fact. All these results will allow us to decompose the operator  $M_N^{\mathcal{P}}$  into two parts  $A_{N,\varepsilon}^{\lambda}$  and  $M_N^{\mathcal{P}} - A_{N,\varepsilon}^{\lambda}$  as was described above and will establish Theorem D: see Section 4. Finally, having proved Theorem D for all  $1 < p \leq \infty$  and Theorem C for  $p = 2$  and  $2 < r < \infty$  we shall employ the interpolation argument from Krause's paper [16] and conclude that Theorem C holds for all  $1 < p < \infty$  and  $r > \max\{p, p/(p-1)\}$ .

**1.1. Notation.** Throughout the whole article, unless otherwise stated, we will write  $A \lesssim B$  ( $A \gtrsim B$ ) if there is an absolute constant  $C > 0$  such that  $A \leq CB$  ( $A \geq CB$ ). Moreover,  $C > 0$  will stand for a large positive constant whose value may vary from occurrence to occurrence. If  $A \lesssim B$  and  $A \gtrsim B$  hold simultaneously then we will write  $A \simeq B$ . Lastly, we will write  $A \lesssim_{\delta} B$  ( $A \gtrsim_{\delta} B$ ) to indicate that the constant  $C > 0$  depends on some  $\delta > 0$ . Let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For a vector  $x \in \mathbb{R}^d$  we set  $|x| = \max\{|x_j| : 1 \leq j \leq d\}$  and  $\mathcal{D} = \{2^n : n \in \mathbb{N}_0\}$  will denote the set of dyadic numbers.

## 2. PRELIMINARIES

**2.1. Variational norm.** Let  $1 \leq r < \infty$ . For each sequence  $(a_j : j \in A)$  where  $A \subseteq \mathbb{Z}$  we define  $r$ -variational seminorm by

$$V_r(a_j : j \in A) = \sup_{\substack{k_0 < k_1 < \dots < k_J \\ k_j \in A}} \left( \sum_{j=1}^J |a_{k_j} - a_{k_{j-1}}|^r \right)^{1/r}.$$

The function  $r \mapsto V_r(a_j : j \in A)$  is non-increasing and satisfies

$$(2.1) \quad \sup_{j \in A} |a_j| \leq V_r(a_j : j \in A) + |a_{j_0}|$$

where  $j_0$  is an arbitrary element of  $A$ . Let

$$\mathcal{V}_r(a_j : j \in A) = \sup_{j \in A} |a_j| + V_r(a_j : j \in A).$$

For any subset  $B \subseteq A$  we have

$$V_r(a_j : j \in B) \leq V_r(a_j : j \in A).$$

Moreover, if  $-\infty \leq u < w < v \leq \infty$  then

$$(2.2) \quad V_r(a_j : u < j < v) \leq 2 \sup_{u < j < v} |a_j| + V_r(a_j : u < j < w) + V_r(a_j : w < j < v).$$

For  $r \geq 2$  we also have

$$(2.3) \quad V_r(a_j : j \in A) \leq 2 \left( \sum_{j \in A} |a_j|^2 \right)^{1/2}.$$

We will need the following simple observation.

**Lemma 1.** *If  $r \geq 2$  then for any sequence  $(a_j : 0 \leq j \leq 2^s)$*

$$(2.4) \quad V_r(a_j : 0 \leq j \leq 2^s) \leq \sqrt{2} \sum_{i=0}^s \left( \sum_{j=0}^{2^{s-i}-1} |a_{(j+1)2^i} - a_{j2^i}|^2 \right)^{1/2}.$$

*Proof.* Let us observe that any interval  $[m, n)$  for  $m, n \in \mathbb{N}$  such that  $0 \leq m < n \leq 2^s$ , is a finite disjoint union of dyadic subintervals, i.e. intervals belonging to some  $\mathcal{I}_i$  for  $0 \leq i \leq s$ , where

$$\mathcal{I}_i = \{[j2^i, (j+1)2^i) : 0 \leq j \leq 2^{s-i} - 1\}$$

and such that each length appears at most twice. For the proof, we set  $m_0 = m$ . Having chosen  $m_p$  we select  $m_{p+1}$  in such a way that  $[m_p, m_{p+1})$  is the longest dyadic interval starting at  $m_p$  and contained inside  $[m_p, n)$ . If the lengths of the selected dyadic intervals increase then we are done. Otherwise, there is  $p$  such that  $m_{p+1} - m_p \geq m_{p+2} - m_{p+1}$ . We show that this implies  $m_{p+2} - m_{p+1} > m_{p+3} - m_{p+2}$ . Suppose for a contradiction that,  $m_{p+2} - m_{p+1} \leq m_{p+3} - m_{p+2}$ . Then

$$[m_{p+1}, 2m_{p+2} - m_{p+1}) \subseteq [m_{p+1}, m_{p+3}).$$

Therefore, it is enough to show that  $2(m_{p+2} - m_{p+1})$  divides  $m_{p+1}$ . It is clear in case  $m_{p+1} - m_p > m_{p+2} - m_{p+1}$ . If  $m_{p+1} - m_p = m_{p+2} - m_{p+1}$  then, by maximality of  $[m_p, m_{p+1})$ ,  $2(m_{p+2} - m_{p+1})$  cannot divide  $m_p$ , thus divides  $m_{p+1}$ .

Next, let  $k_0 < k_1 < \dots < k_J \leq 2^s$  be any increasing sequence. For each  $j \in \{0, \dots, J-1\}$  we may write

$$[k_j, k_{j+1}) = \bigcup_{p=0}^{P_j} [u_p^j, u_{p+1}^j)$$

for some  $P_j \geq 1$  where each interval  $[u_p^j, u_{p+1}^j)$  is dyadic. Then

$$|a_{k_{j+1}} - a_{k_j}| \leq \sum_{p=0}^{P_j} |a_{u_{p+1}^j} - a_{u_p^j}| = \sum_{i=0}^s \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}|.$$

Hence, by Minkowski's inequality

$$\begin{aligned} \left( \sum_{j=0}^{J-1} |a_{k_{j+1}} - a_{k_j}|^2 \right)^{1/2} &\leq \left( \sum_{j=0}^{J-1} \left( \sum_{i=0}^s \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}| \right)^2 \right)^{1/2} \\ &\leq \sum_{i=0}^s \left( \sum_{j=0}^{J-1} \left( \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}| \right)^2 \right)^{1/2}. \end{aligned}$$

Since for a given  $i \in \{0, 1, \dots, 2^s\}$  and  $j \in \{0, 1, \dots, J-1\}$  the inner sums contain at most two elements we obtain

$$\left( \sum_{j=0}^{J-1} |a_{k_{j+1}} - a_{k_j}|^2 \right)^{1/2} \leq \sqrt{2} \sum_{i=0}^s \left( \sum_{j=0}^{J-1} \sum_{p: \{u_p^j, u_{p+1}^j\} \in \mathcal{I}_i} |a_{u_p^j} - a_{u_{p+1}^j}|^2 \right)^{1/2}$$

what is bounded by the right-hand side of (2.4).  $\square$

A long variation seminorm  $V_r^L$  of a sequence  $(a_j : j \in A)$ , is given by

$$V_r^L(a_j : j \in A) = V_r(a_j : j \in A \cap \mathcal{D}).$$

A short variation seminorm  $V_r^S$  is given by

$$V_r^S(a_j : j \in A) = \left( \sum_{n \geq 0} V_r(a_j : j \in A_n)^r \right)^{1/r}$$

where  $A_n = A \cap [2^n, 2^{n+1})$ . Then

$$(2.5) \quad V_r(a_j : j \in \mathbb{N}) \lesssim V_r^L(a_j : j \in \mathbb{N}) + V_r^S(a_j : j \in \mathbb{N}).$$

The next lemma will be used in the estimates for short variations. It illustrates the ideas which have been explored several times (see [14], or recently [16]).

**Lemma 2.** *Let  $u, v \in \mathbb{N}$ ,  $u < v$ . For any integer  $h \in \{1, \dots, v-u\}$  there is a strictly increasing sequence of integers  $(m_j : 0 \leq j \leq h)$  with  $m_0 = u$  and  $m_h = v$  such that for every  $r \geq 2$*

$$V_r(a_j : u \leq j \leq v) \lesssim \left( \sum_{j=0}^h |a_{m_j}|^2 \right)^{1/2} + ((v-u)/h)^{1/2} \left( \sum_{j=u}^{v-1} |a_{j+1} - a_j|^2 \right)^{1/2}.$$

*Proof.* It is enough to consider  $r = 2$ . Fix  $h \in \{1, \dots, v-u\}$  and choose a sequence  $(m_j : 1 \leq j \leq h)$  such that  $m_0 = u$ ,  $m_h = v$  and  $|m_{j+1} - m_j| \simeq (v-u)/h$ . Then

$$\begin{aligned} V_2(a_j : u \leq j \leq v) &\lesssim \left( \sum_{j=0}^h |a_{m_j}|^2 \right)^{1/2} + \left( \sum_{j=0}^{h-1} V_2(a_k : m_j \leq k \leq m_{j+1})^2 \right)^{1/2} \\ &\lesssim \left( \sum_{j=0}^h |a_{m_j}|^2 \right)^{1/2} + \left( \sum_{j=0}^{h-1} \left( \sum_{k=m_j}^{m_{j+1}-1} |a_{k+1} - a_k|^2 \right) \right)^{1/2}. \end{aligned}$$

By the Cauchy–Schwarz inequality the last sum can be dominated by

$$\begin{aligned} &\left( \sum_{j=0}^h |a_{m_j}|^2 \right)^{1/2} + \left( \sum_{j=0}^{h-1} (m_{j+1} - m_j) \sum_{k=m_j}^{m_{j+1}-1} |a_{k+1} - a_k|^2 \right)^{1/2} \\ &\lesssim \left( \sum_{j=0}^h |a_{m_j}|^2 \right)^{1/2} + ((v-u)/h)^{1/2} \left( \sum_{j=u}^{v-1} |a_{j+1} - a_j|^2 \right)^{1/2} \end{aligned}$$

and this completes the proof of the lemma.  $\square$

We observe that, if  $(f_j : j \in \mathbb{N})$  is a sequence of functions in  $\ell^2$  and  $v-u \geq 2$  then

$$(2.6) \quad \|V_r(f_j : j \in [u, v])\|_{\ell^2} \lesssim \max \{A, (v-u)^{1/2} A^{1/2} B^{1/2}\}$$

where

$$A = \max_{u \leq j \leq v} \|f_j\|_{\ell^2}, \quad \text{and} \quad B = \max_{u \leq j < v} \|f_{j+1} - f_j\|_{\ell^2}.$$

Indeed, let

$$h = \lceil (v - u)B / (4A) \rceil.$$

Then  $h \in [1, v - u]$ . If  $h \geq 2$  we may estimate

$$\|V_r(f_j : u \leq j \leq v)\|_{\ell^2} \lesssim Ah^{1/2} + (v - u)B/h^{1/2} \lesssim (v - u)^{1/2} A^{1/2} B^{1/2}$$

where the last inequality follows from Lemma 2. If  $h = 1$  then  $B \lesssim (v - u)^{-1}A$  and hence

$$\|V_r(f_j : u \leq j \leq v)\|_{\ell^2} \lesssim A.$$

**2.2. Lifting lemma.** Let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$  be a mapping whose components  $\mathcal{P}_j$  are integer valued polynomials on  $\mathbb{Z}^k$  such that  $\mathcal{P}_j(0) = 0$ . We set

$$N_0 = \max\{\deg \mathcal{P}_j : 1 \leq j \leq d_0\}.$$

It is convenient to work with the set

$$\Gamma = \{\gamma \in \mathbb{Z}^k \setminus \{0\} : 0 \leq \gamma_j \leq N_0 \text{ for each } j = 1, \dots, k\}$$

with the lexicographic order. Then each  $\mathcal{P}_j$  can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some  $c_j^\gamma \in \mathbb{Z}$ . Let us denote by  $d$  the cardinality of the set  $\Gamma$ . We identify  $\mathbb{R}^d$  with the space of all vectors whose coordinates are labelled by multi-indices  $\gamma \in \Gamma$ . Let  $A$  be a diagonal  $d \times d$  matrix such that

$$(Av)_\gamma = |\gamma|v_\gamma.$$

For  $t > 0$  we set

$$t^A = \exp(A \log t)$$

i.e.  $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$ . Next, we introduce the *canonical* polynomial mapping

$$\mathcal{Q} = (Q_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$$

where  $Q_\gamma(x) = x^\gamma$  and  $x^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$ . The coefficients  $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$  define a linear transformation  $L : \mathbb{R}^d \rightarrow \mathbb{R}^{d_0}$  such that  $L\mathcal{Q} = \mathcal{P}$ . Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each  $j \in \{1, \dots, d_0\}$  and  $v \in \mathbb{R}^d$ . The next lemma, inspired by the continuous analogue (see [7] or [33, p. 515]) reduces proofs of Theorem D and Theorem C to the canonical polynomial mapping.

**Lemma 3.** *Suppose that for some  $p \in (1, \infty)$  and  $r > 2$*

$$\|\mathcal{V}_r(M_N^{\mathcal{Q}} f : N \in \mathbb{N})\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,r} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

*Then*

$$(2.7) \quad \|\mathcal{V}_r(M_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{\ell^p(\mathbb{Z}^{d_0})} \leq C_{p,r} \|f\|_{\ell^p(\mathbb{Z}^{d_0})}.$$

*Proof.* Let  $R > 0$  and  $\Lambda > 0$  be fixed. Let  $f \in \ell^p(\mathbb{Z}^{d_0})$ . In the proof we let  $x \in \mathbb{Z}^{d_0}$ ,  $y \in \mathbb{Z}^k$  and  $u \in \mathbb{Z}^d$ . For any  $x \in \mathbb{Z}^{d_0}$  we define a function  $F_x$  on  $\mathbb{Z}^d$  by

$$F_x(z) = \begin{cases} f(x + L(z)) & \text{if } |z| \leq R + \Lambda^{kN_0}, \\ 0 & \text{otherwise.} \end{cases}$$

If  $|y| \leq N$  and  $|u| \leq R$  then  $|u - \mathcal{Q}(y)| \leq R + \Lambda^{kN_0}$ . Therefore for each  $x \in \mathbb{Z}^{d_0}$

$$M_N^{\mathcal{P}} f(x + Lu) = \frac{1}{N^k} \sum_{y \in \mathbb{N}_N^k} f(x + L(u - \mathcal{Q}(y))) = M_N^{\mathcal{Q}} F_x(u).$$

Hence,

$$\begin{aligned} \|\mathcal{V}_r(M_N^{\mathcal{P}}f : N \in [1, \Lambda])\|_{\ell^p(\mathbb{Z}^{d_0})}^p &= \frac{1}{(2R+1)^d} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{|u| \leq R} \left( \mathcal{V}_r(M_N^{\mathcal{P}}f(x+Lu) : N \in [1, \Lambda]) \right)^p \\ &= \frac{1}{(2R+1)^d} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{|u| \leq R} \left( \mathcal{V}_r(M_N^{\mathcal{Q}}F_x(u) : N \in [1, \Lambda]) \right)^p \leq C_{p,r}^p \frac{1}{R^d} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{u \in \mathbb{Z}^d} |F_x(u)|^p \end{aligned}$$

where in the last inequality we have used

$$\|\mathcal{V}_r(M_N^{\mathcal{Q}}g : N \in [1, \Lambda])\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,r} \|g\|_{\ell^p(\mathbb{Z}^d)}$$

for any  $g \in \ell^p(\mathbb{Z}^d)$ . Since

$$\sum_{x \in \mathbb{Z}^{d_0}} \sum_{u \in \mathbb{Z}^d} |F_x(u)|^p = \sum_{x \in \mathbb{Z}^{d_0}} \sum_{|u| \leq R + \Lambda^{kN_0}} |f(x+Lu)|^p \leq (R + \Lambda^{kN_0})^d \|f\|_{\ell^p(\mathbb{Z}^{d_0})}^p$$

we get

$$\|\mathcal{V}_r(M_N^{\mathcal{P}}f : N \in [1, \Lambda])\|_{\ell^p(\mathbb{Z}^{d_0})}^p \leq C_{p,r}^p \left(1 + \frac{\Lambda^{kN_0}}{R}\right)^d \|f\|_{\ell^p(\mathbb{Z}^{d_0})}^p.$$

Taking  $R$  approaching infinity we conclude

$$\|\mathcal{V}_r(M_N^{\mathcal{P}}f : N \in [1, \Lambda])\|_{\ell^p(\mathbb{Z}^{d_0})}^p \leq C_{p,r}^p \|f\|_{\ell^p(\mathbb{Z}^{d_0})}^p$$

which by monotone convergence theorem implies (2.7).  $\square$

In the rest of the article by  $M_N$  we denote the average for canonical polynomial mapping  $\mathcal{Q}$ , i.e.  $M_N = M_N^{\mathcal{Q}}$ .

**2.3. Gaussian sums.** Given  $q \in \mathbb{N}$  we set  $\mathbb{N}_q = \{a \in \mathbb{N} : 1 \leq a \leq q\}$ . Let  $A_q$  be the subset of  $a \in \mathbb{N}_q^d$  such that

$$\gcd(q, \gcd(a_\gamma : \gamma \in \Gamma)) = 1.$$

For any  $q \in \mathbb{N}$  and  $a \in \mathbb{Z}^d$  we define

$$G(a/q) = q^{-k} \sum_{y \in \mathbb{N}_q^k} e^{2\pi i(a/q) \cdot \mathcal{Q}(y)}.$$

Then there is  $\delta > 0$  such that for any  $a \in A_q$  (see [35, 13])

$$(2.8) \quad |G(a/q)| \lesssim q^{-\delta}.$$

**2.4. Fourier multipliers.** For a function  $f \in L^1(\mathbb{R}^d)$  let

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} f(x) dx$$

be the Fourier transform of  $f$ . If  $f \in \ell^1(\mathbb{Z}^d)$  let

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} e^{2\pi i \xi \cdot x} f(x)$$

be the discrete Fourier transform of  $f$ . For any function  $f : \mathbb{Z}^d \rightarrow \mathbb{C}$  with a finite support we have

$$M_N f(x) = K_N * f(x)$$

where  $K_N$  is a kernel defined by

$$(2.9) \quad K_N(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} \delta_{\mathcal{Q}(y)}$$



and  $\delta_y$  denotes Dirac's delta at  $y \in \mathbb{Z}^k$ . Let  $m_N$  denote the discrete Fourier transform of  $K_N$ , i.e.

$$m_N(\xi) = N^{-k} \sum_{y \in \mathbb{N}_N^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)}.$$

Finally, we define

$$\Phi_N(\xi) = \int_{[0,1]^k} e^{2\pi i \xi \cdot \mathcal{Q}(Ny)} dy.$$

Using a multi-dimensional version of van der Corput lemma (see [33, 6]) we may estimate

$$(2.10) \quad |\Phi_N(\xi)| \lesssim \min \{1, |N^A \xi|^{-1/d}\}.$$

Additionally, we have

$$(2.11) \quad |\Phi_N(\xi) - 1| \lesssim \min \{1, |N^A \xi|\}.$$

### 3. APPROXIMATING MULTIPLIERS

The purpose of this section is to introduce multipliers (3.1), (3.17) and (3.20). In the first two subsections we collect some  $\ell^2(\mathbb{Z}^d)$  and  $\ell^p(\mathbb{Z}^d)$  estimates. Then we apply these results to obtain unrestricted and restricted type inequalities for our multipliers. The last two subsections provide bounds necessary to establish Theorem D and Theorem C. Throughout the rest of the article the maximal functions will be initially defined for any nonnegative finitely supported function  $f$  and unless otherwise stated  $f$  is always such a function.

**3.1.  $\ell^2$ -theory.** We begin with some basic approximations of the multiplier  $m_N$  forced by some multi-dimensional variant of the circle method of Hardy and Littlewood.

We fix  $N \geq 1$ . For any  $\alpha, \beta > 0$  we define a family of *major arcs* by

$$\mathfrak{M}_N = \bigcup_{1 \leq q \leq N^\alpha} \bigcup_{a \in A_q} \mathfrak{M}_N(a/q)$$

where

$$\mathfrak{M}_N(a/q) = \{\xi \in \mathbb{T}^d : |\xi_\gamma - a_\gamma/q| \leq N^{-|\gamma|+\beta} \text{ for all } \gamma \in \Gamma\}.$$

The set  $\mathfrak{m}_N = \mathbb{T}^d \setminus \mathfrak{M}_N$  will be called *minor arc*. We treat the interval  $[0,1]^d$  as  $d$ -dimensional torus  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ .

**Proposition 3.1.** *For any  $\kappa > 0$  there exists  $C > 0$  such that if for some  $1 \leq q \leq N^\alpha$  and  $a \in A_q$*

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq \kappa \cdot N^{-|\gamma|+\beta}$$

*for all  $\gamma \in \Gamma$  then*

$$|m_N(\xi) - G(a/q)\Phi_N(\xi - a/q)| \leq CN^{-1/4},$$

*provided that  $4(\alpha + \beta) < 1$ .*

*Proof.* Let  $\theta = \xi - a/q$ . If  $y, r \in \mathbb{N}^k$  are such that  $y \equiv r \pmod{q}$  then for each  $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q)r^\gamma \pmod{1}.$$

Hence,

$$N^{-k} \sum_{y \in \mathbb{N}_N^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} = N^{-k} \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \sum_{\substack{y \in \mathbb{N}_N^k \\ q|(y-r)}} e^{2\pi i \theta \cdot \mathcal{Q}(y)} = G(a/q)\Phi_N(\xi - a/q) + O(N^{-1/4}).$$

The last equality has been achieved by the mean value theorem, since  $1 \leq q \leq N^\alpha$  and  $|\theta_\gamma| \leq N^{-|\gamma|+\beta}$  for every  $\gamma \in \Gamma$ .  $\square$

For any  $s \in \mathbb{N}$  we set

$$\mathcal{R}_s = \{a/q \in \mathbb{Q}^d : 2^s \leq q < 2^{s+1} \text{ and } a \in A_q\}$$

and  $\mathcal{R}_0 = \{0\}$ . Let  $\nu_N = \sum_{s \geq 0} \nu_N^s$  with a sequence of multipliers  $(\nu_N^s : s \geq 0)$  given by

$$(3.1) \quad \nu_N^s(\xi) = \sum_{a/q \in \mathcal{R}_s} G(a/q) \Phi_N(\xi - a/q) \eta_s(\xi - a/q)$$

where  $\eta_s(\xi) = \eta(10^{(s+1)A}\xi)$  and  $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$  is a smooth function such that  $0 \leq \eta(x) \leq 1$  and

$$(3.2) \quad \eta(x) = \begin{cases} 1 & \text{for } |x| \leq 1/4, \\ 0 & \text{for } |x| \geq 1/2. \end{cases}$$

We may assume that  $\eta$  is a convolution of two smooth nonnegative functions with supports contained in  $[-1/2, 1/2]^d$ .

**Lemma 4.** *If  $4(\alpha + \beta) < 1$  then there are  $C > 0$  and  $\delta_1 > 0$  such that for all  $N \in \mathbb{N}$*

$$\sup_{\xi \in \mathbb{T}^d} |m_N(\xi) - \nu_N(\xi)| \leq CN^{-\delta_1}.$$

*Proof.* Let us notice that for a fixed  $s \in \mathbb{N} \cup \{0\}$  and  $\xi \in \mathbb{T}^d$  the sum (3.1) consists of a single term. Indeed, otherwise there would be different  $a/q, a'/q' \in \mathcal{R}_s$  such that  $\eta_s(\xi - a/q) \neq 0$  and  $\eta_s(\xi - a'/q') \neq 0$ . Thus, for some  $\gamma \in \Gamma$

$$2^{-2s-2} \leq \frac{1}{qq'} \leq \left| \xi_\gamma - \frac{a_\gamma}{q} \right| + \left| \xi_\gamma - \frac{a'_\gamma}{q'} \right| \leq 10^{-s-1}.$$

**Major arcs estimates:** Suppose  $\xi \in \mathfrak{M}_N(a/q)$  with  $1 \leq q \leq N^\alpha$  and  $a \in A_q$ . Let  $s_0$  be such that

$$2^{s_0} \leq q < 2^{s_0+1}.$$

We choose  $s_1 \in \mathbb{N}$  to satisfy

$$2^{s_1+1} \leq N^{1-\alpha-\beta} < 2^{s_1+2}$$

If  $s < s_1$  then for any  $a'/q' \in \mathcal{R}_s$ ,  $a'/q' \neq a/q$  and  $\gamma \in \Gamma$  we have

$$\left| \xi_\gamma - \frac{a'_\gamma}{q'} \right| \geq \frac{1}{qq'} - \left| \xi_\gamma - \frac{a_\gamma}{q} \right| \geq 2^{-s-1}N^{-\alpha} - N^{\beta-|\gamma|} \geq N^{\beta-|\gamma|}.$$

Hence, by (2.10) we get

$$|\Phi_N(\xi - a'/q')| \lesssim |N^A(\xi - a'/q')|^{-1/d} \lesssim N^{-\beta/d}.$$

In particular, by (2.8)

$$(3.3) \quad \left| \sum_{s=0}^{s_1-1} \sum_{\substack{a'/q' \in \mathcal{R}_s \\ a'/q' \neq a/q}} G(a'/q') \Phi_N(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim N^{-\beta/d} \sum_{s=0}^{s_1-1} 2^{-\delta s}.$$

Next, if  $\eta_{s_0}(\xi - a/q) < 1$  then  $|\xi_\gamma - a_\gamma/q| \geq 4^{-1} \cdot 10^{-(s_0+1)|\gamma|}$  for some  $\gamma \in \Gamma$ . Since  $2^{s_0} \leq N^\alpha$ , by (2.10), we get

$$(3.4) \quad |G(a/q) \Phi_N(\xi - a/q)(1 - \eta_{s_0}(\xi - a/q))| \lesssim |N^A(\xi - a/q)|^{-1/d} \lesssim N^{-(1-4\alpha)/d}.$$

Finally, since  $|\Phi_N(\xi)|$  is uniformly bounded, by (2.8) we get

$$(3.5) \quad \left| \sum_{s=s_1}^{\infty} \sum_{\substack{a'/q' \in \mathcal{R}_s \\ a'/q' \neq a/q}} G(a'/q') \Phi_N(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim \sum_{s=s_1}^{\infty} 2^{-\delta s} \lesssim N^{-\delta(1-\alpha-\beta)}.$$

Hence, by Proposition 3.1 and estimates (3.3), (3.4) and (3.5) there exists  $\delta'_1 > 0$  such that for any  $\xi \in \mathfrak{M}_N$

$$(3.6) \quad |m_N(\xi) - \nu_N(\xi)| \lesssim N^{-\delta'_1}.$$

**Minor arcs estimates:**  $\xi \in \mathfrak{m}_N$ . By Dirichlet's principle, for each  $\gamma \in \Gamma$  there are  $1 \leq q_\gamma \leq N^{|\gamma|-\beta}$  and  $(a_\gamma, q_\gamma) = 1$  such that

$$\left| \xi_\gamma - \frac{a_\gamma}{q_\gamma} \right| \leq N^{-|\gamma|+\beta}.$$

Suppose that for all  $\gamma \in \Gamma$ ,  $1 \leq q_\gamma \leq N^{\alpha/d}$ . Then setting  $q = \text{lcm}\{q_\gamma : \gamma \in \Gamma\}$  we have  $q \leq N^\alpha$ . But then  $a \in A_q$  which contradicts to  $\xi \in \mathfrak{m}_N$ . Therefore, there is  $\gamma \in \Gamma$  such that  $N^{\alpha/d} \leq q_\gamma \leq N^{|\gamma|-\beta}$ . By the multi-dimensional version of Weyl's inequality (see [35]), there is  $\delta' > 0$  such that

$$(3.7) \quad |m_N(\xi)| \lesssim N^{-\delta'}.$$

To estimate  $|\nu_N(\xi)|$  we define  $s_1$  by setting

$$2^{s_1} \leq N^\alpha \leq 2^{s_1+1}.$$

If  $s < s_1$  then for any  $a/q \in \mathcal{R}_s$  we have  $q \leq N^\alpha$  and there is  $\gamma \in \Gamma$  such that

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \geq N^{-|\gamma|+\beta}.$$

Thus, by (2.10)

$$|\Phi_N(\xi - a/q)| \lesssim |N^A(\xi - a/q)|^{-1/d} \lesssim N^{-\beta/d}.$$

Hence, by (2.8)

$$(3.8) \quad \left| \sum_{s=0}^{s_1-1} \nu_N^s(\xi) \right| \lesssim N^{-\beta/d} \sum_{s=0}^{\infty} 2^{-\delta s} \lesssim N^{-\beta/d}.$$

For the second part, we proceed similar to (3.5) and obtain

$$(3.9) \quad \left| \sum_{s=s_1}^{\infty} \nu_N^s(\xi) \right| \lesssim \sum_{s=s_1}^{\infty} 2^{-\delta s} \lesssim 2^{-\delta s_1} \lesssim N^{-\alpha/d}.$$

Combining (3.7), (3.8) and (3.9) we can find  $\delta''_1 > 0$  such that for any  $\xi \in \mathfrak{m}_N$

$$(3.10) \quad |m_N(\xi) - \nu_N(\xi)| \lesssim N^{-\delta''_1}.$$

Finally, by (3.6) and (3.10) taking  $\delta_1 = \min\{\delta'_1, \delta''_1\} > 0$  we finish the proof.  $\square$

**3.2.  $\ell^p$ -theory.** Let us recall,  $\eta = \phi * \psi$  for  $\psi, \phi$  smooth nonnegative functions with supports inside  $[-1/2, 1/2]^d$ . The next two lemmas are multi-dimensional analogues of Lemma 1 and Lemma 2 from [25].

**Lemma 5.** For any  $t > 0$  and  $u \in \mathbb{R}^d$

$$(3.11) \quad \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} \eta(t^A \xi) d\xi \right\|_{\ell^1(x)} \leq 1,$$

$$(3.12) \quad \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot u}) \eta(t^A \xi) d\xi \right\|_{\ell^1(x)} \leq |t^{-A} u|.$$

*Proof.* We only show the inequality (3.12) since the proof of (3.11) is almost identical. Let us observe that

$$\eta(t^A \xi) = t^{\text{tr}(A)} \phi_t * \psi_t(\xi)$$

where  $\phi_t(\xi) = \phi(t^A \xi)$ , and  $\psi_t(\xi) = \psi(t^A \xi)$ . For  $x \in \mathbb{Z}^d$  we have

$$t^{-\text{tr}(A)} \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot u}) \eta(t^A \xi) d\xi = \mathcal{F}^{-1} \phi_t(x) \mathcal{F}^{-1} \psi_t(x) - \mathcal{F}^{-1} \phi_t(x - u) \mathcal{F}^{-1} \psi_t(x - u).$$

By Cauchy–Schwarz inequality and Plancherel’s theorem

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} |\mathcal{F}^{-1} \phi_t(x)| |\mathcal{F}^{-1} \psi_t(x) - \mathcal{F}^{-1} \psi_t(x - u)| \\ \leq \|\mathcal{F}^{-1} \phi_t\|_{\ell^2} \left\| \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot u}) \psi_t(\xi) d\xi \right\|_{\ell^2(x)} = \|\phi_t\|_{L^2} \|(1 - e^{2\pi i \xi \cdot u}) \psi_t(\xi)\|_{L^2(d\xi)}. \end{aligned}$$

Moreover, since

$$\int_{\mathbb{R}^d} |1 - e^{-2\pi i \xi \cdot u}|^2 |\psi_t(\xi)|^2 d\xi \lesssim |t^{-A} u|^2 \int_{\mathbb{R}^d} |t^A \xi|^2 |\psi_t(\xi)|^2 d\xi \lesssim t^{-\text{tr}(A)} |t^{-A} u|^2 \int_{\mathbb{R}^d} |\xi|^2 |\psi(\xi)|^2 d\xi$$

we obtain

$$\sum_{x \in \mathbb{Z}^d} |\mathcal{F}^{-1} \phi_t(x)| |\mathcal{F}^{-1} \psi_t(x) - \mathcal{F}^{-1} \psi_t(x - u)| \leq 2^{-1} t^{-\text{tr}(A)} |t^{-A} u| \|\phi\|_{L^2} \|\psi\|_{L^2}$$

which finishes the proof of (3.12) since  $\|\phi\|_{L^2} \|\psi\|_{L^2} \leq 1$ .  $\square$

**Proposition 3.2.** *For each  $p \in (1, \infty)$ ,  $r > 2$  and  $t > 0$  we have*

$$\|\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \eta(t^A \cdot) \hat{f}) : N \in \mathbb{N})\|_{\ell^p} \leq C_{p,r} \|\mathcal{F}^{-1}(\eta(t^A \cdot) \hat{f})\|_{\ell^p}.$$

*Proof.* Let  $\varrho_t(\xi) = \eta(t^A \xi)$ . Since  $\varrho_t = \varrho_t \varrho_{t/2}$  by Hölder’s inequality we have

$$\begin{aligned} |\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}))(x) : N \in \mathbb{N}|^p &\leq \left( \int_{\mathbb{R}^d} \mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}))(u) : N \in \mathbb{N} |\mathcal{F}^{-1} \varrho_{t/2}(x - u)| du \right)^p \\ &\leq \int_{\mathbb{R}^d} \left( \mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}))(u) : N \in \mathbb{N} \right)^p |\mathcal{F}^{-1} \varrho_{t/2}(x - u)| du \cdot \|\mathcal{F}^{-1} \varrho_{t/2}\|_{L^1}^{p-1}. \end{aligned}$$

Next, we note that  $\|\mathcal{F}^{-1} \varrho_{t/2}\|_{L^1} \lesssim 1$  and

$$\sum_{x \in \mathbb{Z}^d} |\mathcal{F}^{-1} \varrho_{t/2}(x - u)| \lesssim t^{-\text{tr}(A)} \sum_{x \in \mathbb{Z}^d} \frac{1}{1 + |t^A(x - u)|^2}$$

which is uniformly bounded with respect to  $A$ . Thus we obtain

$$\|\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}) : N \in \mathbb{N})\|_{\ell^p} \lesssim \|\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}) : N \in \mathbb{N})\|_{L^p} \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{L^p}$$

where the last inequality is a consequence of Remark after Theorem 1.5 in [15, p. 6717]. The proof will be completed if we show

$$\|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{L^p} \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}.$$

For this purpose we use (3.12) from Lemma 5. We have

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} \int_{[0,1]^d} |\mathcal{F}^{-1}(\varrho_t \hat{f})(x + u) - \mathcal{F}^{-1}(\varrho_t \hat{f})(x)|^p du \\ \leq \int_{[0,1]^d} \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{-2\pi i \xi \cdot u}) \varrho_{t/2}(\xi) d\xi \right\|_{\ell^1(x)}^p \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}^p du \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}^p. \end{aligned}$$

Hence,

$$\|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{L^p}^p = \sum_{x \in \mathbb{Z}^d} \int_{[0,1]^d} |\mathcal{F}^{-1}(\varrho_t \hat{f})(x+u)|^p du \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}^p.$$

This finishes the proof.  $\square$

For  $t \in \mathbb{N}_0$  we set  $Q_t = (2^{t+1})!$  and define

$$\varrho_t(\xi) = \eta(Q_{t+1}^{3dA} \xi).$$

**Lemma 6.** *Let  $p \in [1, \infty)$  and  $m \in \mathbb{N}_{Q_t}^d$ . Then for any  $t \in \mathbb{N}_0$  we have*

$$\|\mathcal{F}^{-1}(\varrho_t \hat{f})(Q_t x + m)\|_{\ell^p(x)} \simeq Q_t^{-d/p} \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}.$$

*Proof.* For each  $m \in \mathbb{N}_{Q_t}^d$  we set

$$J_m = \|\mathcal{F}^{-1}(\varrho_t \hat{f})(Q_t x + m)\|_{\ell^p(x)},$$

and  $I = \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}$ . Then

$$\sum_{m \in \mathbb{N}_{Q_t}^d} J_m^p = I^p.$$

Since  $\varrho_t = \varrho_t \varrho_{t-1}$ , by Minkowski's inequality we obtain that

$$\begin{aligned} & \|\mathcal{F}^{-1}(\varrho_t \hat{f})(Q_t x + m) - \mathcal{F}^{-1}(\varrho_t \hat{f})(Q_t x + m')\|_{\ell^p(x)} \\ &= \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot (Q_t x + m)} (1 - e^{2\pi i \xi \cdot (m - m')}) \varrho_t(\xi) \hat{f}(\xi) d\xi \right\|_{\ell^p(x)} \\ &\leq \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot (m - m')}) \varrho_{t-1}(\xi) d\xi \right\|_{\ell^1(x)} I \leq |Q_t^{-3dA}(m - m')| I \end{aligned}$$

where in the last step we have used Lemma 5. Hence, for all  $m, m' \in \mathbb{N}_{Q_t}^d$

$$J_{m'} \leq J_m + Q_t^{1-3d} I \leq J_m + Q_t^{-2d} I.$$

Thus

$$(3.13) \quad J_{m'}^p \leq 2^{p-1} J_m^p + 2^{p-1} Q_t^{-2dp} I^p.$$

Therefore,

$$I^p = \sum_{m' \in \mathbb{N}_{Q_t}^d} J_{m'}^p \leq 2^{p-1} Q_t^d J_m^p + 2^{p-1} Q_t^{d(1-2p)} I^p.$$

By the definition of  $Q_t$  we have

$$2^p Q_t^{d(1-2p)} \leq 1.$$

Hence, we obtain  $I^p \leq 2^p Q_t^d J_m^p$ . For the converse inequality, we use again (3.13) to get

$$Q_t^d J_{m'}^p \leq 2^{p-1} \sum_{m \in \mathbb{N}_{Q_t}^d} J_m^p + 2^{p-1} Q_t^{d(1-2p)} I^p \leq 2^p I^p.$$

$\square$

**Proposition 3.3.** *Let  $p \in (1, \infty)$ ,  $r > 2$  and  $m \in \mathbb{N}_{Q_t}^k$ . Then for any  $t \in \mathbb{N}_0$  we have*

$$\|\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f})(Q_t x + m) : N \in \mathbb{N})\|_{\ell^p(x)} \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})(Q_t x + m)\|_{\ell^p(x)}.$$

*Proof.* For each  $m \in \mathbb{N}_{Q_t}^d$  we define

$$J_m = \|\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}))(Q_t x + m) : N \in \mathbb{N}\|_{\ell^p(x)}.$$

Then, by Proposition 3.2,

$$I^p = \sum_{m \in \mathbb{N}_{Q_t}^d} J_m^p = \|\mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}) : N \in \mathbb{N})\|_{\ell^p}^p \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}^p.$$

If  $m, m' \in \mathbb{N}_{Q_t}^d$  then we may write

$$\begin{aligned} & \left\| \mathcal{V}_r \left( \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot (Q_t x + m)} (1 - e^{2\pi i \xi \cdot (m - m')}) \Phi_N(\xi) \varrho_t(\xi) \hat{f}(\xi) d\xi : N \in \mathbb{N} \right) \right\|_{\ell^p(x)} \\ & \lesssim \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{-2\pi i \xi \cdot (m - m')}) \varrho_t(\xi) \hat{f}(\xi) d\xi \right\|_{\ell^p(x)}. \end{aligned}$$

Since  $\varrho_t = \varrho_t \varrho_{t-1}$ , by Minkowski's inequality and Lemma 5 the last expression may be dominated by

$$\left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{-2\pi i \xi \cdot (m - m')}) \varrho_{t-1}(\xi) \hat{f}(\xi) d\xi \right\|_{\ell^1(x)} \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p} \leq Q_t^{-2d} \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p},$$

thus

$$J_m \leq J_{m'} + Q_t^{-2d} \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}.$$

Raising to  $p$ 'th power and summing up over  $m' \in \mathbb{N}_{Q_t}^d$  we get

$$Q_t^d J_m^p \leq 2^{p-1} I^p + 2^{p-1} Q_t^{d(1-2p)} \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}^p \lesssim \|\mathcal{F}^{-1}(\varrho_t \hat{f})\|_{\ell^p}^p$$

and Lemma 6 finishes the proof.  $\square$

**3.3. Unrestricted inequalities.** We start by proving  $\ell^2(\mathbb{Z}^d)$ -boundedness of  $r$ -variations for  $\nu_{2^j}$ . The proofs of the estimates like in (3.14) are based on Bourgain's 'logarithmic' type lemmas (see [4], see also [16, 17]). We present different approach, based on a direct analysis of this multiplier. For the proof of Theorem D in case  $p = 2$  we use Theorem 1.

**Theorem 1.** *For any  $r > 2$ , there are  $\delta_2 > 0$  and  $C > 0$  such that for any  $s \in \mathbb{N}_0$  and  $f \in \ell^2(\mathbb{Z}^d)$*

$$(3.14) \quad \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : j \geq 0) \right\|_{\ell^2} \leq C 2^{-s\delta_2} \|f\|_{\ell^2}.$$

*Proof.* The proof will consist of two parts where we shall estimate separately the pieces of  $r$ -variations where  $0 \leq j \leq 2^{\kappa_s}$  and  $j \geq 2^{\kappa_s}$  where  $\kappa_s = 20d(s+1)$ . By (2.2) and (2.1) we see that

$$(3.15) \quad \begin{aligned} \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : j \geq 0) \right\|_{\ell^2} & \lesssim \left\| \mathcal{F}^{-1}(\nu_{2^{2\kappa_s}}^s \hat{f}) \right\|_{\ell^2} + \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : 0 \leq j \leq 2^{\kappa_s}) \right\|_{\ell^2} \\ & \quad + \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : j \geq 2^{\kappa_s}) \right\|_{\ell^2}. \end{aligned}$$

By Plancherel's theorem, (2.8) and the disjointness of supports of  $\eta_s(\xi - a/q)$ 's while  $a/q$  varies over  $\mathcal{R}_s$ , the first term in (3.15) is bounded by  $2^{-s\delta} \|f\|_{\ell^2}$ . To estimate the second term we apply Lemma 1. Indeed, let  $I_j^i = [j2^i, (j+1)2^i)$  and note that (2.4) and Plancherel's theorem give

$$\begin{aligned} \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : 0 \leq j \leq 2^{\kappa_s}) \right\|_{\ell^2} & \lesssim \sum_{i=0}^{\kappa_s} \left\| \left( \sum_{j=0}^{2^{\kappa_s-i}-1} \left| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\nu_{2^{m+1}}^s \hat{f}) - \mathcal{F}^{-1}(\nu_{2^m}^s \hat{f}) \right|^2 \right)^{1/2} \right\|_{\ell^2} \\ & = \sum_{i=0}^{\kappa_s} \left( \sum_{j=0}^{2^{\kappa_s-i}-1} \int_{\mathbb{T}^d} \left| \sum_{m \in I_j^i} (\nu_{2^{m+1}}^s(\xi) - \nu_{2^m}^s(\xi)) \hat{f}(\xi) \right|^2 d\xi \right)^{1/2}. \end{aligned}$$

Next, for any  $i \in \{0, 1, \dots, \kappa_s\}$  we have

$$\begin{aligned} \sum_{j=0}^{2^{\kappa_s-i}-1} \int_{\mathbb{T}^d} \left| \sum_{m \in I_j^i} \nu_{2^{m+1}}^s(\xi) - \nu_{2^m}^s(\xi) \right|^2 |\hat{f}(\xi)|^2 d\xi \\ \leq \sum_{j=0}^{2^{\kappa_s-i}-1} \int_{\mathbb{T}^d} \sum_{m, m' \in I_j^i} |\nu_{2^{m+1}}^s(\xi) - \nu_{2^m}^s(\xi)| \cdot |\nu_{2^{m'+1}}^s(\xi) - \nu_{2^{m'}}^s(\xi)| \cdot |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

Let  $\Delta_m(\xi) = \Phi_{2^{m+1}}(\xi) - \Phi_{2^m}(\xi)$ . Using (2.10) and (2.11) we can estimate

$$\sum_{m \in \mathbb{N}} |\Delta_m(\xi)| \lesssim \sum_{m \in \mathbb{N}} \min \{ |2^{mA}\xi|, |2^{mA}\xi|^{-1/d} \} \lesssim 1.$$

Therefore, by the disjointness of supports of  $\eta_s(\cdot - a/q)$ 's we obtain

$$\begin{aligned} \sum_{j=0}^{2^{\kappa_s-i}-1} \int_{\mathbb{T}^d} \left| \sum_{m \in I_j^i} \nu_{2^{m+1}}^s(\xi) - \nu_{2^m}^s(\xi) \right|^2 |\hat{f}(\xi)|^2 d\xi \\ \leq \sum_{a/q \in \mathcal{R}_s} |G(a/q)|^2 \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m, m' \in I_j^i} \int_{\mathbb{T}^d} |\Delta_m(\xi - a/q)| \cdot |\Delta_{m'}(\xi - a/q)| \cdot \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \\ \lesssim \sum_{a/q \in \mathcal{R}_s} |G(a/q)|^2 \int_{\mathbb{T}^d} \left( \sum_{j=0}^{\infty} \min \{ |2^{jA}(\xi - a/q)|, |2^{jA}(\xi - a/q)|^{-1/d} \} \right)^2 \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \end{aligned}$$

which, by (2.8), is bounded by  $2^{-2s\delta} \|f\|_{\ell^2}^2$ . Finally, it remains to estimate the last term in (3.15). Let us observe that if  $x \in \mathbb{Z}^d$  then

$$\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f})(x) = \sum_{a/q \in \mathcal{R}_s} e^{-2\pi i(a/q) \cdot x} G(a/q) \mathcal{F}^{-1}(\Phi_{2^j} \eta_s \hat{f}(\cdot + a/q))(x).$$

For any  $x, y \in \mathbb{Z}^d$  we set

$$I(x, y) = V_r \left( \sum_{a/q \in \mathcal{R}_s} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\Phi_{2^j} \eta_s \hat{f}(\cdot + a/q))(y) : j \geq 2^{\kappa_s} \right).$$

and

$$J(x, y) = \sum_{a/q \in \mathcal{R}_s} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\eta_s \hat{f}(\cdot + a/q))(y).$$

We notice, functions  $x \mapsto I(x, y)$  and  $x \mapsto J(x, y)$  are  $Q_s \mathbb{Z}^d$ -periodic. If  $u \in \mathbb{N}_{Q_s}^d$  and  $a/q \in \mathcal{R}_s$ , by Plancherel's theorem we get

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\Phi_{2^j} \eta_s \hat{f}(\cdot + a/q))(x + u) - \mathcal{F}^{-1}(\Phi_{2^j} \eta_s \hat{f}(\cdot + a/q))(x) \right\|_{\ell^2(x)} \\ = \left\| (1 - e^{-2\pi i \xi \cdot u}) \Phi_{2^j}(\xi) \eta_s(\xi) \hat{f}(\xi + a/q) \right\|_{L^2(d\xi)} \lesssim 2^{-j/d} |u| \cdot \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{L^2} \end{aligned}$$

since, for  $\xi \in \mathbb{T}^d$

$$|\xi| |\Phi_{2^j}(\xi)| \lesssim |\xi| |2^{jA}\xi|^{-1/d} \lesssim 2^{-j/d}.$$

Therefore, by the triangle inequality, (2.3) and Plancherel's theorem

$$\left| \|I(x, x + u)\|_{\ell^2(x)} - \|I(x, x)\|_{\ell^2(x)} \right| \leq Q_s \sum_{j \geq 2^{\kappa_s}} \sum_{a/q \in \mathcal{R}_s} 2^{-j/d} \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{L^2}.$$

Since  $\mathcal{R}_s$  contains at most  $2^{s(d+1)}$  rational numbers we have

$$\sum_{a/q \in \mathcal{R}_s} \|\eta_s(\cdot - a/q)\hat{f}\|_{L^2} \leq 2^{(d+1)s} \|f\|_{\ell^2}.$$

Hence, using  $2^{\kappa_s}/d - (s+1)2^{s+1} - (d+1)s \geq \delta s$  we obtain

$$\|I(x, x)\|_{\ell^2(x)} \lesssim \|I(x, x+u)\|_{\ell^2(x)} + 2^{-\delta s} \|f\|_{\ell^2}.$$

Thus, we may estimate

$$(3.16) \quad \left\| V_r(\mathcal{F}^{-1}(\nu_j^s \hat{f}) : j \geq 2^{\kappa_s}) \right\|_{\ell^2}^2 \lesssim \frac{1}{Q_s^d} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x+u)\|_{\ell^2(x)}^2 + 2^{-2\delta s} \|f\|_{\ell^2}^2.$$

Next, by double change of variables and periodicity we get

$$\sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x+u)\|_{\ell^2(x)}^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(x-u, x)^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(u, x)^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2$$

what, using Proposition 3.2 and (2.8), is bounded by

$$\begin{aligned} \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(u, x)\|_{\ell^2(x)}^2 &= \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(x, x+u)\|_{\ell^2(x)}^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \int_{\mathbb{T}^d} \left| \sum_{a/q \in \mathcal{R}_s} G(a/q) e^{2\pi i(a/q) \cdot u} \eta_s(\xi - a/q) \hat{f}(\xi) \right|^2 d\xi \\ &\lesssim 2^{-2\delta s} Q_s^d \cdot \|f\|_{\ell^2}^2. \end{aligned}$$

Finally, combining with (3.16) we obtain an estimate on the last term in (3.15).  $\square$

**Theorem 2.** *There exists  $C > 0$  such that for every  $f \in \ell^2(\mathbb{Z}^d)$*

$$\left\| \sup_{N \in \mathbb{N}} |M_N f| \right\|_{\ell^2} \leq C \|f\|_{\ell^2}.$$

*Proof.* In view of (2.1) it suffices to apply Theorem 1 and Lemma 4.  $\square$

For each  $N \in \mathbb{N}$  and  $t \in \mathbb{N}_0$  we define new multipliers

$$(3.17) \quad \Omega_N^t(\xi) = \sum_{a \in \mathbb{N}_{Q_t}^d} G(a/Q_t) \Phi_N(\xi - a/Q_t) \varrho_t(\xi - a/Q_t).$$

Then

**Theorem 3.** *Let  $p \in (1, \infty)$  and  $r > 2$ . There exists  $C_{p,r} > 0$  such that for any  $t \in \mathbb{N}_0$*

$$\left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Omega_N^t \hat{f}) : N \in \mathbb{N}) \right\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

*Proof.* Let us observe that

$$\mathcal{F}^{-1}(\Phi_N(\cdot - a/Q_t) \varrho_t(\cdot - a/Q_t) \hat{f})(Q_t x + m) = \mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}(\cdot + a/Q_t))(Q_t x + m) e^{-2\pi i(a/Q_t) \cdot m}.$$

Therefore,

$$\left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Omega_N^t \hat{f}) : N \in \mathbb{N}) \right\|_{\ell^p}^p = \sum_{m \in \mathbb{N}_{Q_t}^d} \left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t \hat{f}(\cdot + a/Q_t))(Q_t x + m) : N \in \mathbb{N}) \right\|_{\ell^p(x)}^p$$

where

$$(3.18) \quad F(\xi; m) = \sum_{a \in \mathbb{N}_{Q_t}^d} G(a/Q_t) \hat{f}(\xi + a/Q_t) e^{-2\pi i(a/Q_t) \cdot m}.$$



Now, by Proposition 3.3 and the definition (3.18) we get

$$\begin{aligned} & \sum_{m \in \mathbb{N}_{Q_t}^d} \left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Phi_N \varrho_t F(\cdot; m))(Q_t x + m) : N \in \mathbb{N}) \right\|_{\ell^p(x)}^p \\ & \lesssim \sum_{m \in \mathbb{N}_{Q_t}^d} \left\| \mathcal{F}^{-1}(\varrho_t F(\cdot; m))(Q_t x + m) \right\|_{\ell^p(x)}^p = \left\| \sum_{a \in \mathbb{N}_{Q_t}^d} G(a/Q_t) \mathcal{F}^{-1}(\varrho_t(\cdot - a/Q_t) \hat{f}) \right\|_{\ell^p}^p. \end{aligned}$$

Using Minkowski's inequality we may estimate

$$\left\| \sum_{a \in \mathbb{N}_{Q_t}^d} G(a/Q_t) \mathcal{F}^{-1}(\varrho_t(\cdot - a/Q_t) \hat{f}) \right\|_{\ell^p} \leq \left\| \sum_{a \in \mathbb{N}_{Q_t}^d} e^{-2\pi i(a/Q_t) \cdot x} G(a/Q_t) \mathcal{F}^{-1} \varrho_t(x) \right\|_{\ell^1(x)} \|f\|_{\ell^p}.$$

We notice that for  $x \in \mathbb{Z}^d$  we have

$$\sum_{a \in \mathbb{N}_{Q_t}^d} e^{-2\pi i(a/Q_t) \cdot x} = \begin{cases} Q_t^d & \text{if } Q_t \mid x, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if  $x \equiv m \pmod{Q_t}$  then

$$\sum_{a \in \mathbb{N}_{Q_t}^d} e^{-2\pi i(a/Q_t) \cdot x} G(a/Q_t) = Q_t^{-k} \sum_{y \in \mathbb{N}_{Q_t}^k} \sum_{a \in \mathbb{N}_{Q_t}^d} e^{2\pi i(a/Q_t) \cdot (\mathcal{Q}(y) - m)} = Q_t^{d-k} L_m$$

where  $L_m = \#\{y \in \mathbb{N}_{Q_t}^k : \mathcal{Q}(y) \equiv m \pmod{Q_t}\}$ . Let us observe that  $L_m \cap L_{m'} = \emptyset$  if  $m \neq m'$ . Now, by Lemma 6

$$\begin{aligned} \left\| \sum_{a \in \mathbb{N}_{Q_t}^d} e^{-2\pi i(a/Q_t) \cdot x} G(a/Q_t) \mathcal{F}^{-1} \varrho_t(x) \right\|_{\ell^1(x)} &= Q_t^{d-k} \sum_{m \in \mathbb{N}_{Q_t}^d} L_m \cdot \left\| \mathcal{F}^{-1} \varrho_t(Q_t x + m) \right\|_{\ell^1(x)} \\ &\lesssim Q_t^{-k} \sum_{m \in \mathbb{N}_{Q_t}^d} L_m \cdot \left\| \mathcal{F}^{-1} \varrho_t \right\|_{\ell^1} \lesssim 1 \end{aligned}$$

what together with Lemma 5 finishes the proof.  $\square$

**3.4. Restricted inequalities.** This subsection is devoted to study certain multipliers with  $r$ -variations restricted to large and small cubes, i.e. when the side length of cubes in our averages is small or large. Let us define

$$(3.19) \quad \kappa_t = 20d(t+1).$$

**3.4.1. Large cubes.** For any  $t \in \mathbb{N}_0$  we will consider a variational norm for averages over cubes with sides bigger than  $2^{2^{\kappa_t}}$ . First, let us define auxiliary multipliers for each  $N \in \mathbb{N}$  and  $t \in \mathbb{N}_0$  by

$$(3.20) \quad \Lambda_N^t(\xi) = \sum_{a/q \in \mathcal{Q}_t} G(a/q) \Phi_N(\xi - a/q) \varrho_t(\xi - a/q)$$

where

$$\mathcal{Q}_t = \{a/q \in \mathbb{Q}^d : q \geq 2^{t+1}, q \mid Q_t \text{ and } a \in A_q\}.$$

We show

**Proposition 3.4.** *Let  $r > 2$ . There are  $\delta_3 > 0$  and  $C_r > 0$  such that for any  $t \in \mathbb{N}_0$  and  $f \in \ell^2(\mathbb{Z}^d)$*

$$\left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Lambda_{2^j}^t \hat{f}) : j \geq 2^{\kappa_t}) \right\|_{\ell^2} \leq C_r 2^{-t\delta_3} \|f\|_{\ell^2}.$$

*Proof.* We notice that

$$\mathcal{F}^{-1}(\Lambda_{2^j}^t \hat{f})(Q_t x + m) = \mathcal{F}^{-1}(\Phi_{2^j} \varrho_t F(\cdot; m))(Q_t x + m)$$

where

$$F(\xi; m) = \sum_{a/q \in \mathcal{Q}_t} G(a/q) \hat{f}(\xi + a/q) e^{-2\pi i(a/q) \cdot m}.$$

By Proposition 3.3 and Lemma 6 we get

$$\begin{aligned} \left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Lambda_{2^j}^t \hat{f}) : j \geq 2^{\kappa_t}) \right\|_{\ell^2}^2 &= \sum_{m \in \mathbb{N}_{Q_t}^d} \left\| \mathcal{V}_r(\mathcal{F}^{-1}(\Lambda_{2^j}^t \hat{f})(Q_t x + m) : j \geq 2^{\kappa_t}) \right\|_{\ell^2(x)}^2 \\ &\lesssim \sum_{m \in \mathbb{N}_{Q_t}^d} \left\| \mathcal{F}^{-1}(\varrho_t F(\cdot; m))(Q_t x + m) \right\|_{\ell^2(x)}^2 = \left\| \sum_{a/q \in \mathcal{Q}_t} G(a/q) \mathcal{F}^{-1}(\varrho_t(\cdot - a/q) \hat{f}) \right\|_{\ell^2}^2. \end{aligned}$$

Using Plancherel's theorem we may write

$$\left\| \sum_{a/q \in \mathcal{Q}_t} G(a/q) \mathcal{F}^{-1}(\varrho_t(\cdot - a/q) \hat{f}) \right\|_{\ell^2}^2 = \sum_{a/q \in \mathcal{Q}_t} |G(a/q)|^2 \left\| \varrho_t(\cdot - a/q) \hat{f} \right\|_{L^2}^2$$

which, by (2.8), is bounded by  $2^{-2\delta t} \|f\|_{\ell^2}^2$ .  $\square$

Finally, we show

**Theorem 4.** *Let  $r > 2$ . There are  $\delta_4 > 0$  and  $C_r > 0$  such that for any  $t \in \mathbb{N}_0$  and  $f \in \ell^2(\mathbb{Z}^d)$*

$$\left\| \mathcal{V}_r(M_{2^j} f - \mathcal{F}^{-1}(\Omega_{2^j}^t \hat{f}) : j > 2^{\kappa_t}) \right\|_{\ell^2} \leq C_r 2^{-\delta_4 t} \|f\|_{\ell^2}.$$

*Proof.* Let us notice

$$\Omega_{2^j}^t(\xi) = \sum_{s=0}^t \sum_{a/q \in \mathcal{R}_s} G(a/q) \Phi_{2^j}(\xi - a/q) \varrho_t(\xi - a/q) + \Lambda_{2^j}^t(\xi)$$

and observe that

$$m_{2^j}(\xi) - \Omega_{2^j}^t(\xi) = \left( m_{2^j}(\xi) - \sum_{s \geq 0} \nu_{2^j}^s(\xi) \right) + \left( \sum_{s=0}^t \nu_{2^j}^s(\xi) - \Omega_{2^j}^t(\xi) - \Lambda_{2^j}^t(\xi) \right) + \sum_{s > t} \nu_{2^j}^s(\xi) + \Lambda_{2^j}^t(\xi).$$

The last two terms are covered by Theorem 1 and Proposition 3.4 respectively, whereas the first term is bounded thanks to Lemma 4 since  $j \geq 2^{\kappa_t}$ . Thus it remains to estimate the second term. First, we observe that  $\varrho_t(\xi - a/q) - \eta_s(\xi - a/q) \neq 0$  implies that there is  $\gamma \in \Gamma$  such that  $|\xi_\gamma - a_\gamma/q| \geq 4^{-1} Q_t^{-3d|\gamma|}$ . Therefore, for  $j \geq 2^{\kappa_t}$

$$2^{j|\gamma|} \cdot \left| \xi_\gamma - \frac{a_\gamma}{q} \right| \gtrsim 2^j Q_t^{-3d^2} \gtrsim 2^{j/2},$$

and using (2.10), we get

$$|\Phi_{2^j}(\xi - a/q)| \lesssim 2^{-j/(2d)}.$$

Finally, by (2.8) we obtain

$$\left| \Omega_{2^j}^t(\xi) - \Lambda_{2^j}^t(\xi) - \sum_{s=0}^t \nu_{2^j}^s(\xi) \right| \leq \sum_{s=0}^t 2^{-\delta s} |\Phi_{2^j}(\xi - a/q)| \cdot |\varrho_t(\xi - a/q) - \eta_s(\xi - a/q)| \lesssim 2^{-j/(2d)}.$$

This completes the proof of Theorem 4.  $\square$

3.4.2. *Small cubes.* Theorem 5 will be the main result of this subsection. The proof will be based on ideas of Bourgain [4]. Bourgain used this restricted type maximal function with logarithmic loss to obtain the full range of  $p \in (1, \infty)$  in his maximal theorem.

**Theorem 5.** *For every  $p \in (1, \infty]$  there exists a constant  $C_p > 0$  such that for all  $J \in \mathbb{N}$*

$$(3.21) \quad \left\| \sup_{J < j \leq 2J} |K_{2^j} * f| \right\|_{\ell^p} \leq C_p (\log J) \|f\|_{\ell^p},$$

for every  $f \in \ell^p(\mathbb{Z}^d)$ .

*Proof.* Since we are working with the averaging operator it suffices to prove (3.21) for  $p \in (1, 2]$  and nonnegative function  $f$ . Let  $\tilde{K}_m(x) = K_m(-x)$ . By the duality, for every  $x \in \mathbb{Z}^d$ , there is a sequence of nonnegative numbers  $(g_j(x) : J < j \leq 2J)$  such that  $\sum_{J < j \leq 2J} g_j(x) \leq 1$  and

$$\left\| \sup_{J < j \leq 2J} |K_{2^j} * f| \right\|_{\ell^p} \leq 2 \left\| \sum_{J < j \leq 2J} (K_{2^j} * f) g_j \right\|_{\ell^p} \leq 2 \sup_{\|h\|_{\ell^r} \leq 1} \left\| \sum_{J < j \leq 2J} \tilde{K}_{2^j} * (h g_j) \right\|_{\ell^r} \|f\|_{\ell^p},$$

where  $r = p/(p-1) \geq 2$ . Therefore, it suffices to prove that for every  $p \in (1, 2]$  with an integer  $r = p/(p-1)$  and any finite  $F \subseteq \mathbb{Z}^d$  we have

$$\left\| \sum_{J < j \leq 2J} \tilde{K}_{2^j} * h_j \right\|_{\ell^r} \leq C_r (\log J) |F|^{1/r}$$

where  $h_j = g_j \mathbf{1}_F \geq 0$ .

We partition the set  $(J, 2J] \cap \mathbb{Z}$  into at most  $2\mu(\log_2 J)$  subsets  $S$  with the sparseness property

$$(3.22) \quad l, l' \in S, \quad \text{if } l \neq l' \quad \text{then } |l - l'| \geq 2\mu(\log_2 J)$$

where  $\mu > 0$  is a constant satisfying (3.34). Therefore, it is enough to prove that for each integer  $r \geq 2$

$$(3.23) \quad \left\| \sum_{j \in S} \tilde{K}_{2^j} * h_j \right\|_{\ell^r} \leq C_r |F|^{1/r}.$$

We show (3.23) by induction with respect to  $r$ . For  $r = 2$  we have

$$\left\| \sum_{j \in S} \tilde{K}_{2^j} * h_j \right\|_{\ell^2} \leq |F|^{1/2} \sup_{\|f\|_{\ell^2} = 1} \left\| \sum_{j \in S} (K_{2^j} * f) g_j \right\|_{\ell^2} \leq |F|^{1/2} \sup_{\|f\|_{\ell^2} = 1} \left\| \sup_{j \in S} K_{2^j} * f \right\|_{\ell^2} \lesssim |F|^{1/2}$$

where in the last step we have used Theorem 2. For  $r > 2$  we expand the left-hand side of (3.23). There is a constant  $C_r > 0$ , which may depend only on  $r$  and such that

$$(3.24) \quad \left\| \sum_{j \in S} \tilde{K}_{2^j} * h_j \right\|_{\ell^r}^r \leq C_r \sum_{J < j_1 < \dots < j_r \leq 2J} \sum_{x \in \mathbb{Z}^d} \prod_{n=1}^r \tilde{K}_{2^{j_n}} * h_{j_n}(x) + C_r \sum_{x \in \mathbb{Z}^d} \left( \sum_{j \in S} \tilde{K}_{2^j} * h_j(x) \right)^{r-1}.$$

To treat the first term in (3.24) we need to prove that for any increasing sequence  $J = j_0 < j_1 < \dots < j_r \leq 2J$

$$(3.25) \quad \left\| \left( \prod_{n=2}^r \tilde{K}_{2^{j_n}} * h_{j_n} \right) * (K_{2^{j_1}} - K_{2^{j_0}}) \right\|_{\ell^2} \lesssim J^{-r} |F|^{1/2}$$

Assuming momentarily (3.25) we would have

$$\left| \sum_{x \in \mathbb{Z}^d} (\tilde{K}_{2^{j_1}} - \tilde{K}_{2^{j_0}}) * h_{j_1}(x) \prod_{n=2}^r \tilde{K}_{2^{j_n}} * h_{j_n}(x) \right| \leq |F|^{1/2} \left\| \left( \prod_{n=2}^r \tilde{K}_{2^{j_n}} * h_{j_n} \right) * (K_{2^{j_1}} - K_{2^{j_0}}) \right\|_{\ell^2} \lesssim J^{-r} |F|,$$

thus

$$\begin{aligned}
(3.26) \quad & \sum_{J < j_1 < \dots < j_r \leq 2J} \sum_{x \in \mathbb{Z}^d} \prod_{n=1}^r \tilde{K}_{2^{j_n}} * h_{j_n}(x) \\
& \leq |F| + \sum_{J < j_1 < \dots < j_r \leq 2J} \sum_{x \in \mathbb{Z}^d} \tilde{K}_{2^{j_0}} * h_{j_1}(x) \prod_{n=2}^r \tilde{K}_{2^{j_n}} * h_{j_n}(x) \\
& \leq |F| + \sum_{x \in \mathbb{Z}^d} \left( \tilde{K}_{2^{j_0}} * \sum_{j \in S} h_j \right)(x) \left( \sum_{j \in S} \tilde{K}_{2^j} * h_j(x) \right)^{r-1} \\
& \leq |F| + \sum_{x \in \mathbb{Z}^d} \left( \sum_{j \in S} \tilde{K}_{2^j} * h_j(x) \right)^{r-1}.
\end{aligned}$$

Hence, by the inductive hypothesis we obtain

$$\left\| \sum_{j \in S} \tilde{K}_{2^j} * h_j \right\|_{\ell^r}^r \lesssim |F| + \left\| \sum_{j \in S} \tilde{K}_{2^j} * h_j \right\|_{\ell^{r-1}}^{r-1} \lesssim |F|.$$

This completes the proof and shows that (3.23) holds.

It remains to prove the bound (3.25). First, we introduce approximating multipliers

$$\Upsilon_N^t(\xi) = \sum_{s=0}^t \sum_{a/q \in \mathcal{R}_s} G(a/q) \Phi_N(\xi - a/q) \eta_s(\xi - a/q) \varrho_N(\xi - a/q)$$

where  $\varrho_N(\xi) = \eta((J^{-\mu}N)^A \xi)$ . Then, by Lemma 4 and estimates (2.8) and (2.10)

$$(3.27) \quad |m_N(\xi) - \Upsilon_N^t(\xi)| \lesssim N^{-\delta_1} + 2^{-t\delta} + J^{-\mu/d}.$$

Moreover,

$$(3.28) \quad \|\mathcal{F}^{-1}(\Upsilon_N^t)\|_{\ell^1} \lesssim 2^{t(d+1)}.$$

For each  $n \in \mathbb{N}$  we set

$$t_n = \max \{r, (2d+3)\delta^{-1}\}^n (\log_2 J)$$

By Plancherel's theorem and (3.27) we have

$$(3.29) \quad \|\tilde{K}_{2^{j_n}} * h_{j_n} - \mathcal{F}^{-1}(\Upsilon_{2^{j_n}}^{t_n} \hat{h}_{j_n})\|_{\ell^2} \lesssim (2^{-j_n \delta_1} + 2^{-t_n \delta} + J^{-\mu/d}) |F|^{1/2}.$$

Moreover, by (3.28), for every  $x \in \mathbb{Z}^d$  we have

$$(3.30) \quad |\mathcal{F}^{-1}(\Upsilon_{2^{j_n}}^{t_n} \hat{h}_{j_n})(x)| \leq \|\mathcal{F}^{-1}(\Upsilon_{2^{j_n}}^{t_n})\|_{\ell^1} \leq C 2^{t_n(d+1)}.$$

Let us denote by  $\mathcal{W}$  the support of  $(\Upsilon_{2^{j_2}}^{t_2} \hat{h}_{j_2}) * \dots * (\Upsilon_{2^{j_r}}^{t_r} \hat{h}_{j_r})$ . Then

$$\mathcal{W} \subseteq \bigcup_{q=1}^{2^{rt_r}} \bigcup_{a \in \mathbb{N}_q^d} \{\xi \in \mathbb{T}^d : |\xi_\gamma - a_\gamma/q| < 2^{-j_2|\gamma|} J^{\mu|\gamma|} \text{ for all } \gamma \in \Gamma\}.$$

Furthermore, by Hölder's inequality and (3.30) we have

$$\begin{aligned}
(3.31) \quad & \left\| \prod_{n=2}^r \mathcal{F}^{-1}(\Upsilon_{2^{j_n}}^{t_n} \hat{h}_{j_n}) \right\|_{\ell^2} \leq \prod_{n=2}^r \|\mathcal{F}^{-1}(\Upsilon_{2^{j_n}}^{t_n} \hat{h}_{j_n})\|_{\ell^{2(r-1)}} \leq \prod_{n=2}^r \|\mathcal{F}^{-1}(\Upsilon_{2^{j_n}}^{t_n})\|_{\ell^1} \prod_{n=2}^r \|h_{j_n}\|_{\ell^{2(r-1)}} \\
& \lesssim 2^{(t_2 + \dots + t_r)(d+1)} |F|^{1/2} \lesssim 2^{2t_r(d+1)} |F|^{1/2}
\end{aligned}$$

because  $t_2 + \dots + t_r \leq 2t_r$ . Next, by (3.22) we have  $j_2 - j_1 \geq 2\mu(\log_2 J)$  thus

$$J^\mu 2^{-j_2} \leq 2^{-j_1}.$$

In particular, if  $\xi \in \mathcal{W}$  then there are  $1 \leq q \leq 2^{rt_r}$  and  $a \in \mathbb{N}_q^d$  such that for each  $\gamma \in \Gamma$  we have

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq J^{\mu|\gamma|} 2^{-j_2|\gamma|} \leq 2^{-j_1(|\gamma|-\beta)}.$$

Since  $2^{rt_r} \leq 2^{\alpha J}$  for sufficiently large  $J$ , by Proposition 3.1 and (2.11), we have

$$(3.32) \quad \begin{aligned} |m_{2^{j_1}}(\xi) - m_{2^{j_0}}(\xi)| &\leq |\Phi_{2^{j_1}}(\xi - a/q) - 1| + |\Phi_{2^{j_0}}(\xi - a/q) - 1| + 2^{-j_0/4} + 2^{-j_1/4} \\ &\lesssim |2^{j_1 A}(\xi - a/q)| + |2^{j_0 A}(\xi - a/q)| + 2^{-J/4} \lesssim J^{-\mu}. \end{aligned}$$

Next, we may estimate

$$(3.33) \quad \begin{aligned} &\left\| \left( \prod_{n=2}^r \tilde{K}_{2^{j_n}} * h_{j_n} \right) * (K_{2^{j_1}} - K_{2^{j_0}}) \right\|_{\ell^2} \\ &\leq \sum_{n=2}^r \prod_{k=2}^{n-1} \left\| \mathcal{F}^{-1}(\tilde{\Upsilon}_{2^{j_k}}^{t_k} \hat{h}_{j_k}) \right\|_{\ell^\infty} \cdot \left\| \tilde{K}_{2^{j_n}} * h_{j_n} - \mathcal{F}^{-1}(\tilde{\Upsilon}_{2^{j_n}}^{t_n} \hat{h}_{j_n}) \right\|_{\ell^2} \\ &\quad + \left\| \prod_{n=2}^r \mathcal{F}^{-1}(\tilde{\Upsilon}_{2^{j_n}}^{t_n} \hat{h}_{j_n}) \right\|_{\ell^2} \cdot \sup_{\xi \in \mathcal{W}} |m_{2^{j_1}}(\xi) - m_{2^{j_0}}(\xi)|. \end{aligned}$$

By (3.29) and (3.30) the first term in (3.33) is bounded by

$$\begin{aligned} &\sum_{n=2}^r \left( \prod_{k=2}^{n-1} 2^{(d+1)t_k} \right) (2^{-\delta t_n} + 2^{-j_n \delta_1} + J^{-\mu/d}) |F|^{1/2} \\ &\lesssim \sum_{n=2}^r 2^{2(d+1)t_{n-1} - \delta t_n} |F|^{1/2} + 2^{2(d+1)t_r} (2^{-J\delta_1} + J^{-\mu/d}) |F|^{1/2} \\ &\lesssim 2^{-t_1} |F|^{1/2} + 2^{2(d+1)t_r} (2^{-J\delta_1} + J^{-\mu/d}) |F|^{1/2}. \end{aligned}$$

Moreover, by (3.31) and (3.32) the second term in (3.33) is bounded by  $2^{2(d+1)t_r} J^{-\mu} |F|^{1/2}$ . Since  $2^{-t_1} \leq J^{-r}$  it is enough to select  $\mu$  satisfying

$$(3.34) \quad \mu \geq 2(d+1)^3 \max \{r, (2d+3)\delta^{-1}\}^r.$$

This completes the proof of Theorem 5.  $\square$

#### 4. MAXIMAL THEOREM

We are ready to proof Theorem D. In view of Lemma 3 it is enough to show

**Theorem 6.** *Let  $p \in (1, \infty)$ . There is  $C_p > 0$  such that for every  $f \in \ell^p(\mathbb{Z}^d)$*

$$(4.1) \quad \left\| \sup_{N \in \mathbb{N}} |M_N^Q f| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

Let us observe that the supremum in (4.1) may be restricted to the set of dyadic numbers  $\mathcal{D}$ . As we mentioned in the introduction we shall explore restricted interpolation lemma of Ionescu and Wainger introduced in [13] (see also [11]). Namely,

**Lemma 7.** *Suppose for each  $r \in (1, 2]$ ,  $\epsilon \in (0, 1]$  and  $\lambda > 0$  there is a sequence of linear operators  $(A_j^{\lambda, \epsilon} : j \in \mathbb{N})$  such that*

$$\left\| \sup_{j \in \mathbb{N}} |A_j^{\lambda, \epsilon} f| \right\|_{\ell^r} \leq C_{\epsilon, r} \lambda^\epsilon \|f\|_{\ell^r} \quad \text{and} \quad \left\| \sup_{j \in \mathbb{N}} |M_{2^j} f - A_j^{\lambda, \epsilon} f| \right\|_{\ell^2} \leq D_\epsilon \lambda^{-1} \|f\|_{\ell^2}.$$

Then for each  $p \in (1, 2]$  there exists a constant  $C_p > 0$

$$\left\| \sup_{j \in \mathbb{N}} |M_{2^j} f| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

*Proof of Theorem 6.* Let  $\epsilon \in (0, 1]$ . If  $\lambda \leq 1$  then we may take  $A_j^{\lambda, \epsilon} = 0$  since by Theorem 2

$$\left\| \sup_{j \in \mathbb{N}} |M_{2^j} f - A_j^{\lambda, \epsilon} f| \right\|_{\ell^2} \leq C \lambda^{-1} \|f\|_{\ell^2}.$$

For  $\lambda > 1$  we choose  $t \in \mathbb{N}$  such that

$$t = \lfloor \delta_4^{-1} \log_2 \lambda \rfloor + 1$$

where  $\delta_4 > 0$  is the exponent from Theorem 4. Let  $\kappa_t$  be defined by (3.19). If  $j < 2^{\kappa_t}$  then we set  $A_j^{\lambda, \epsilon} = M_{2^j}$ . By Theorem 5 we may write

$$\left\| \sup_{j < 2^{\kappa_t}} |M_{2^j} f| \right\|_{\ell^r} \leq \sum_{k=1}^{\kappa_t} \left\| \sup_{2^{k-1} \leq j < 2^k} |M_{2^j} f| \right\|_{\ell^r} \lesssim \sum_{k=1}^{\kappa_t} k \cdot \|f\|_{\ell^r} \lesssim t^2 \cdot \|f\|_{\ell^r} \lesssim \lambda^\epsilon \|f\|_{\ell^r}.$$

For  $j \geq 2^{\kappa_t}$  we define  $A_j^{\lambda, \epsilon} = \Omega_{2^j}^t$ . Then by Theorem 3 and Theorem 4 we have

$$\left\| \sup_{j \geq 2^{\kappa_t}} |\mathcal{F}^{-1}(\Omega_{2^j}^t \hat{f})| \right\|_{\ell^r} \lesssim \|f\|_{\ell^r} \quad \text{and} \quad \left\| \sup_{j \geq 2^{\kappa_t}} |M_{2^j} f - \mathcal{F}^{-1}(\Omega_{2^j}^t \hat{f})| \right\|_{\ell^2} \lesssim \lambda^{-1} \|f\|_{\ell^2}.$$

This completes the proof of Theorem 6.  $\square$

## 5. VARIATIONAL THEOREM

In this section we prove Theorem C. Again, using Lemma 3 it is enough show

**Theorem 7.** *Let  $p \in (1, \infty)$  and  $r > \max\{p, p/(p-1)\}$ . There is  $C_{p,r} > 0$  such that for each  $f \in \ell^p(\mathbb{Z}^d)$*

$$(5.1) \quad \left\| V_r(M_N^Q f : N \in \mathbb{N}) \right\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

*Proof.* We only prove (5.1) for  $p = 2$ . In order to obtain (5.1) for  $p \in (1, \infty)$  and  $r > \max\{p, p/(p-1)\}$  it suffices to repeat the argument from [16], and interpolate the estimate (4.1) with the estimate (5.1) for  $p = 2$ . To prove the inequality (5.1) for  $p = 2$ , we will make use of the estimate (2.5) and separately treat long and short variations.

**Long variations:** In order to estimate long variations we shall use Theorem 1. Indeed, for  $r > 2$

$$\begin{aligned} \left\| V_r^L(M_N f : N \in \mathbb{N}) \right\|_{\ell^2} &= \left\| V_r(M_{2^j} f : j \geq 0) \right\|_{\ell^2} \\ &\leq \sum_{s \geq 0} \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : j \geq 0) \right\|_{\ell^2} + \left\| V_r\left(M_{2^j} f - \sum_{s \geq 0} \mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : j \geq 0\right) \right\|_{\ell^2} \\ &\lesssim \sum_{s \geq 0} \left\| V_r(\mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) : j \geq 0) \right\|_{\ell^2} + \left( \sum_{j \geq 0} \left\| M_{2^j} f - \sum_{s \geq 0} \mathcal{F}^{-1}(\nu_{2^j}^s \hat{f}) \right\|_{\ell^2}^2 \right)^{1/2} \\ &\lesssim \sum_{s \geq 0} 2^{-s\delta_2} \|f\|_{\ell^2} + \sum_{j \geq 0} 2^{-j\delta_1} \|f\|_{\ell^2} \end{aligned}$$

where the penultimate inequality follows from (2.3) and the last inequality is guaranteed by Theorem 1 and Lemma 4.

**Short variations:** Let us define the Fourier projection  $\Pi_Q$  onto a set  $Q \subseteq \mathbb{T}^d$  by setting

$$\Pi_Q f = \mathcal{F}^{-1}(\mathbb{1}_Q \hat{f})$$

and observe that according to the definition of short variations we have

$$(5.2) \quad \begin{aligned} \left\| V_r^S(M_N f : N \in \mathbb{N}) \right\|_{\ell^2}^2 &\lesssim \sum_{n \geq 0} \left\| V_2(M_N(\Pi_{\mathfrak{M}_{2^n}} f) : 2^n \leq N < 2^{n+1}) \right\|_{\ell^2}^2 \\ &\quad + \sum_{n \geq 0} \left\| V_2(M_N(\Pi_{\mathfrak{m}_{2^n}} f) : 2^n \leq N < 2^{n+1}) \right\|_{\ell^2}^2 \end{aligned}$$

for major  $\mathfrak{M}_{2^j}$  and minor  $\mathfrak{m}_{2^j}$  arcs defined in Section 3. The proof of Theorem 7 will be completed if we show that the sums in (5.2) can be dominated by  $\|f\|_{\ell^2}^2$ . Applying (2.6) we get the desired bound for the second sum in (5.2). Indeed, by Plancherel's theorem and Weyl's inequality [35], we have

$$\|M_N(\Pi_{\mathfrak{m}_{2^n}} f)\|_{\ell^2} \leq \sup_{\xi \in \mathfrak{m}_{2^n}} |m_N(\xi)| \cdot \|\Pi_{\mathfrak{m}_{2^n}} f\|_{\ell^2} \lesssim 2^{-n\delta'} \|\Pi_{\mathfrak{m}_{2^n}} f\|_{\ell^2}$$

and

$$\|M_{N+1}(\Pi_{\mathfrak{m}_{2^n}} f) - M_N(\Pi_{\mathfrak{m}_{2^n}} f)\|_{\ell^2} \lesssim 2^{-n} \|\Pi_{\mathfrak{m}_{2^n}} f\|_{\ell^2}.$$

Therefore, using inequality (2.6) we obtain

$$\sum_{n \geq 0} \|V_2(M_N(\Pi_{\mathfrak{m}_{2^n}} f) : 2^n \leq N < 2^{n+1})\|_{\ell^2}^2 \lesssim \sum_{n \geq 0} 2^{-n\delta'} \|\Pi_{\mathfrak{m}_{2^n}} f\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2.$$

To deal with the first sum in (5.2) we need to have a more subtle decomposition of the family of major arcs. For  $u \geq -\beta n$  let us define

$$\mathfrak{N}_{2^n}^u(a/q) = \mathfrak{M}_{2^n}^u(a/q) \setminus \mathfrak{M}_{2^n}^{u+1}(a/q)$$

where  $\mathfrak{M}_{2^n}^u(a/q) = \{\xi \in \mathbb{T}^d : |\xi_\gamma - a_\gamma/q| \leq 2^{-n|\gamma|-u} \text{ for all } \gamma \in \Gamma\}$ . Setting

$$\mathfrak{N}_{2^n,s}^u = \bigcup_{a/q \in \mathcal{R}_s} \mathfrak{N}_{2^n}^u(a/q).$$

we may write

$$(5.3) \quad \mathfrak{M}_{2^n} = \bigcup_{0 \leq s \leq \alpha n - 1} \bigcup_{a/q \in \mathcal{R}_s} \mathfrak{M}_{2^n}(a/q) = \bigcup_{0 \leq s \leq \alpha n - 1} \bigcup_{u \geq -\beta n} \mathfrak{N}_{2^n,s}^u.$$

Let  $2^n \leq N < 2^{n+1}$ . If  $\xi \in \mathfrak{M}_{2^n} \cap \mathfrak{N}_{2^n}^u(a/q)$  for  $a/q \in \mathcal{R}_s$  then for all  $\gamma \in \Gamma$

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq 2^d N^{-|\gamma|+\beta}.$$

Thus, by Proposition 3.1 and (2.8) together with (2.10) and (2.11), we get

$$(5.4) \quad |m_N(\xi) - m_{2^n}(\xi)| \lesssim 2^{-n/4} + 2^{-\delta s} |\Phi_N(\xi - a/q) - \Phi_{2^n}(\xi - a/q)| \lesssim 2^{-\delta s} (2^{-|u|/d} + 2^{-n/8})$$

provided that  $8\alpha\delta \leq 1$ . Next, we set

$$\tilde{\mathfrak{N}}_{2^n,s}^u = \bigcup_{u > dn/8} \mathfrak{N}_{2^n,s}^u$$

and observe that, by (5.3), we get

$$\begin{aligned} \|V_r(M_N(\Pi_{\mathfrak{M}_{2^n}} f) : 2^n \leq N < 2^{n+1})\|_{\ell^2} &= \|V_r(\mathcal{F}^{-1}((m_N - m_{2^n})\mathbb{1}_{\mathfrak{M}_{2^n}} \hat{f}) : 2^n \leq N < 2^{n+1})\|_{\ell^2} \\ &\lesssim \sum_{0 \leq s \leq \alpha n - 1} \sum_{-\beta n \leq u \leq dn/8} \|V_2(\mathcal{F}^{-1}((m_N - m_{2^n})\mathbb{1}_{\mathfrak{N}_{2^n,s}^u} \hat{f}) : 2^n \leq N < 2^{n+1})\|_{\ell^2} \\ &\quad + \sum_{0 \leq s \leq \alpha n - 1} \|V_2(\mathcal{F}^{-1}((m_N - m_{2^n})\mathbb{1}_{\tilde{\mathfrak{N}}_{2^n,s}^u} \hat{f}) : 2^n \leq N < 2^{n+1})\|_{\ell^2}. \end{aligned}$$

If  $-\beta n \leq u \leq dn/8$  then using (2.6) and (5.4) we can estimate

$$\|V_2(\mathcal{F}^{-1}((m_N - m_{2^n})\mathbb{1}_{\mathfrak{N}_{2^n,s}^u} \hat{f}) : 2^n \leq N < 2^{n+1})\|_{\ell^2} \lesssim 2^{-\delta s/2} 2^{-|u|/(2d)} \|\Pi_{\mathfrak{N}_{2^n,s}^u} f\|_{\ell^2},$$

otherwise

$$\|V_2(\mathcal{F}^{-1}((m_N - m_{2^n})\mathbb{1}_{\tilde{\mathfrak{N}}_{2^n,s}^u} \hat{f}) : 2^n \leq N < 2^{n+1})\|_{\ell^2} \lesssim 2^{-\delta s/2} 2^{-n/16} \|\Pi_{\tilde{\mathfrak{N}}_{2^n,s}^u} f\|_{\ell^2}.$$

Therefore, by Cauchy–Schwarz inequality we get

$$\begin{aligned} \|V_r(M_N(\Pi_{\mathfrak{M}_{2^n}} f) : 2^n \leq N < 2^{n+1})\|_{\ell^2}^2 &\lesssim \sum_{0 \leq s \leq \alpha n - 1} 2^{-\delta s/2} \sum_{u \geq -\beta n} 2^{-|u|/(2d)} \|\Pi_{\mathfrak{M}_{2^n, s}^u} f\|_{\ell^2}^2 \\ &\quad + 2^{-n/8} \sum_{s \geq 0} 2^{-\delta s/2} \|\Pi_{\tilde{\mathfrak{M}}_{2^n, s}} f\|_{\ell^2}^2. \end{aligned}$$

Next, we have

$$\begin{aligned} \sum_{n \geq 0} \sum_{0 \leq s \leq \alpha n - 1} 2^{-\delta s/2} \sum_{u \geq -\beta n} 2^{-|u|/(2d)} \|\Pi_{\mathfrak{M}_{2^n, s}^u} f\|_{\ell^2}^2 &\lesssim \sum_{s \geq 0} 2^{-\delta s/2} \sum_{n \geq s/\alpha} \sum_{u \geq -\beta n} 2^{-|u|/(2d)} \|\Pi_{\mathfrak{M}_{2^n, s}^u} f\|_{\ell^2}^2 \\ &\lesssim \sum_{s \geq 0} 2^{-\delta s/2} \sum_{u \in \mathbb{Z}} 2^{-|u|/(2d)} \sum_{n \geq \max\{s/\alpha, -u/\beta\}} \|\Pi_{\mathfrak{M}_{2^n, s}^u} f\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2 \end{aligned}$$

where the last inequality follows since  $\mathfrak{M}_{2^n, s}^u$  are disjoint for  $n \geq \max\{s/\alpha, -u/\beta\}$  while  $u \in \mathbb{Z}$  and  $s \in \mathbb{N}_0$  are fixed. Hence,

$$\sum_{n \geq 0} \|V_r(M_N(\Pi_{\mathfrak{M}_{2^n}} f) : 2^n \leq N < 2^{n+1})\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2 + \sum_{n \geq 0} 2^{-n/8} \|f\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2.$$

This provides the bound for the second sum in (5.2) and completes the proof of Theorem 7 for  $p = 2$  and  $r > 2$ .  $\square$

## REFERENCES

- [1] M. Boshernitzan, G. Kolesnik, A. Quas, and M. Wierdl, *Ergodic averaging sequences*, J. Anal. Math. **95** (2005), no. 1, 63–103.
- [2] J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers*, Israel J. Math. **61** (1988), 39–72.
- [3] ———, *On the pointwise ergodic theorem on  $L^p$  for arithmetic sets*, Israel J. Math. **61** (1988), 73–84.
- [4] ———, *Pointwise ergodic theorems for arithmetic sets. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein.*, Publ. Math.-Paris (1989), no. 69, 5–45.
- [5] Z. Buczolich and R. D. Mauldin, *Divergent square averages*, Ann. Math. **171** (2010), no. 3, 1479–1530.
- [6] A. Carbery, M. Christ, and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. (1999), no. 4, 981–1015.
- [7] K. deLeeuw, *On  $L^p$  multipliers*, Ann. Math. **81** (1965), 364–379.
- [8] C. Demeter, M. T. Lacey, T. Tao, and Ch. Thiele, *Breaking the duality in the return times theorem*, Duke Math. J. **143** (2008), no. 2, 281–355.
- [9] K. Hughes, *Problems and results related to Waring’s problem: Maximal functions and ergodic averages*, arXiv:1310.7904, 2013.
- [10] A. D. Ionescu, *An endpoint estimate for the discrete spherical maximal function*, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1411–1417.
- [11] A. D. Ionescu, A. Magyar, E. M. Stein, and S. Wainger, *Discrete Radon transforms and applications to ergodic theory*, Acta Math.-Djursholm (2007), no. 198, 231 – 298.
- [12] A. D. Ionescu, A. Magyar, and S. Wainger, *Averages along polynomial sequences in discrete nilpotent Lie groups: Singular Radon transforms*, Advances in Analysis: The Legacy of Elias M. Stein (Ch. Fefferman, A. D. Ionescu, D. H. Phong, and S. Wainger, eds.), Princeton University Press, 2014, p. 480.
- [13] A. D. Ionescu and S. Wainger,  *$L^p$  boundedness of discrete singular Radon transforms*, J. Amer. Math. Soc. **19** (2006), no. 2, 357–383.
- [14] R. L. Jones, R. Kaufman, J. M. Rosenblatt, and M. Wierdl, *Oscillation in ergodic theory*, Ergod. Theor. Dyn. Syst. **18** (1998), no. 4, 889–935.
- [15] R. L. Jones, A. Seeger, and J. Wright, *Strong variational and jump inequalities in harmonic analysis*, Trans. Amer. Math. Soc. (2008), 6711–6742.
- [16] B. Krause, *Polynomial ergodic averages converge rapidly: Variations on a theorem of Bourgain*, arXiv:1402.1803, 2014.
- [17] ———, *Some optimizations for (maximal) multipliers in  $l^p$* , arXiv:1402.1804, 2014.
- [18] M. T. Lacey, *On an inequality due to Bourgain*, Ill. J. Math. **41** (1997), no. 2, 231–236.
- [19] ———, *The bilinear maximal functions map into  $l^p$  for  $2/3 < p \leq 1$* , Ann. Math. **151** (2000), 35–57.
- [20] P. LaVictoire, *Universally  $L^1$ -bad arithmetic sequences*, J. Anal. Math. **113** (2011), no. 1, 241–263.



- [21] A. Magyar, E. M. Stein, and S. Wainger, *Discrete analogues in harmonic analysis: Spherical averages*, Ann. Math. (2002), 189–208.
- [22] ———, *Maximal operators associated to discrete subgroups of nilpotent Lie groups*, J. Anal. Mat. **101** (2007), no. 1, 257–312.
- [23] M. Mirek,  *$\ell^p(\mathbb{Z})$  - boundedness of discrete maximal functions along thin subsets of primes and pointwise ergodic theorems*, Accepted for publication in Math. Z., 2013.
- [24] ———, *Weak type  $(1, 1)$  inequalities for discrete rough maximal functions*, Accepted for publication in J. Anal. Mat., 2013.
- [25] M. Mirek and Trojan B., *Cotlar's ergodic theorem along the prime numbers*, arXiv:1311.7572, 2014.
- [26] F. Nazarov, R. Oberlin, and Ch. Thiele, *A Calderón Zygmund decomposition for multiple frequencies and an application to an extension of a lemma of Bourgain*, Math. Res. Lett. **17** (2010), no. 3, 529–545.
- [27] D. M. Oberlin, *Two discrete fractional integrals*, Math. Res. Lett. **8** (2001), 1–6.
- [28] R. Oberlin, *Estimates for compositions of maximal operators with singular integrals*, Canad. Math. Bull. **56** (2013), no. 4, 801–813.
- [29] R. Oberlin, A. Seeger, T. Tao, Ch. Thiele, and J. Wright, *A variation norm Carleson theorem*, J. Eur. Math. Soc. **14** (2012), no. 2, 421–464.
- [30] L. B. Pierce, *Discrete analogues in harmonic analysis*, Ph.D. thesis, Princeton University, 2009.
- [31] ———, *Discrete fractional Radon transforms and quadratic forms*, Duke Math. J. **161** (2012), no. 1, 69–106.
- [32] ———, *A note on twisted discrete singular Radon transforms*, Int. Math. Res. Not. IMRN (2012), no. 1, 17–33.
- [33] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.
- [34] E. M. Stein and S. Wainger, *Discrete analogues of singular Radon transforms*, Bull. Amer. Math. Soc. (1990), no. 23, 537 – 544.
- [35] ———, *Discrete analogues in harmonic analysis, I:  $\ell^2$  estimates for singular Radon transforms*, Amer. J. Math. **121** (1999), no. 6, 1291–1336.
- [36] ———, *Discrete analogues in harmonic analysis II: fractional integration*, J. Anal. Math. **80** (2000), no. 1, 335–355.
- [37] ———, *Two discrete fractional integral operators revisited*, J. Anal. Math. **87** (2002), no. 1, 451–479.
- [38] Ch. Thiele, *The maximal quartile operator*, Rev. Mat. Iberoam. **17** (2001), no. 1, 107–136.
- [39] M. Wierdl, *Pointwise ergodic theorem along the prime numbers*, Israel J. Math. **64** (1988), no. 3, 315–336.
- [40] P. Zorin-Kranich, *Variation estimates for averages along primes and polynomials*, arXiv:1403.4085, 2014.

MARIUSZ MIREK, UNIVERSITÄT BONN, MATHEMATICAL INSTITUTE, ENDENICHER ALLEE 60, D-53115 BONN, GERMANY & INSTYTUT MATEMATYCZNY, UNIwersYTET WROCLAWSKI, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND  
*E-mail address:* mirek@math.uni-bonn.de

BARTOSZ TROJAN, INSTYTUT MATEMATYCZNY, UNIwersYTET WROCLAWSKI, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND  
*E-mail address:* trojan@math.uni.wroc.pl